Dynamical Stability of Six-dimensional Warped Flux Compactification

Hiroyuki Yoshiguchi[a], Shinji Mukohyama[a,b],
Yuuiti Sendouda[a], Shunichiro Kinoshita[a]

[a]*Department of Physics*
The University of Tokyo, Tokyo 113-0033, Japan

[b]*Research Center for the Early Universe*
The University of Tokyo, Tokyo 113-0033, Japan

Abstract

We show the dynamical stability of a six-dimensional braneworld solution with warped flux compactification recently found by the authors. We consider linear perturbations around this background spacetime, assuming the axisymmetry in the extra dimensions. The perturbations are expanded by scalar-, vector- and tensor-type harmonics of the four-dimensional Minkoswki spacetime and we analyze each type separately. It is found that there is no unstable mode in each sector and that there are zero modes only in the tensor sector, corresponding to the four-dimensional gravitons. We also obtain the first few Kaluza-Klein modes in each sector.

arXiv:hep-th/0512212v2 8 Mar 2006

1 Introduction

It is being established that there are two major accelerating, or quasi de Sitter, phases in the history of our universe. One is inflation in the early universe and the other is the current phase of accelerating expansion. This picture is supported by observations such as cosmic microwave background and supernovae [1, 2, 3], and thus we are almost sure that there are and were such accelerating phases. However, we essentially do not know what causes those phases, i.e. we do not yet know what inflaton and dark energy are from the viewpoint of fundamental physics.

In string theory it had been thought rather difficult to construct a four-dimensional de Sitter universe with stabilized moduli until the construction by Kachru, Kallosh, Linde and Trivedi (KKLT) [4]. (See [5, 6] for followup proposals.) They evaded previously known no-go theorems by putting anti-D-branes at the bottom of a warped throat after stabilizing all moduli. Since the shape of the warped, compact extra dimensions are stabilized by fluxes, this set-up is often called warped flux compactification. This set-up provides a number of possible applications to cosmology [7, 8, 9, 10, 11].

In our previous paper [12] we pointed out that brane gravity in the warped flux compactification is somehow similar to that in the first Randall-Sundrum (RS1) scenario [13] with radion stabilization [14]. (See refs. [15, 16, 17, 18] for brane gravity in the RS1 scenario with radion stabilization.) The evolution of matter on the brane changes the bulk geometry not only near the brane but possibly everywhere in the whole extra dimensions. Provided that all moduli are stabilized, the bulk geometry should quickly settle to a configuration which is determined by the boundary condition, i.e. the brane source(s), values of conserved quantities and the regularity of the other region of the extra dimensions. As a consequence of the change of the bulk geometry, the induced geometry on the brane responds to the evolution of the matter source on the brane. The four-dimensional Einstein theory is recovered as a rather indirect and subtle relation between the matter source on the brane and the response of the induced geometry. To support this picture, we considered a simplified situation in which we can see the recovery of the four-dimensional Einstein theory in the warped flux compactification. In particular, we found an exact solution representing a six-dimensional brane world with warped flux compactification, including a warped geometry, compactification, a magnetic flux, and one or two 3-brane(s). (See refs. [19, 20, 21, 22, 23, 24] for other models of six-dimensional brane world.)

The purpose of this paper is to show the stability of the six-dimensional exact solution found in the previous paper [12]. For simplicity we set the four-dimensional

cosmological constant to zero and assume axisymmetry in the extra dimensions. We expand linear perturbations by scalar-, vector- and tensor-type harmonics of the four-dimensional Minkowski spacetime and analyze each type separately. Linear perturbations in six-dimensional models are also considered in [25, 26]. We shall show that there is no unstable mode in each sector and that there are zero modes only in the tensor sector, corresponding to the four-dimensional gravitons. We also obtain the first few Kaluza-Klein modes in each sector.

The rest of this paper is organized as follows. In section 2, we briefly review our six-dimensional brane world model. We then consider linear perturbations with axisymmetry to show the stability of our exact solution in section 3. We numerically solve the perturbed Einstein equations and Maxwell equations for each type of perturbations, and then show that there is no unstable mode. In section 4, we summarize the main results and discuss them.

2 6D Warped Flux Compactification

In this section, we briefly review our six-dimensional braneworld model, which captures essential features of the warped flux compactification, that is, warped geometry, magnetic flux of an antisymmetric field along the extra dimensions, and branes. The model is simple enough to make it possible for us to analyze gravity on the brane from higher dimensional point of view. Our start point is the six-dimensional Einstein-Maxwell action,

$$I_6 = \frac{M_6^4}{2} \int d^6x \sqrt{-g} \Big(R - 2\Lambda_6 - \frac{1}{2} F^{MN} F_{MN} \Big), \tag{2.1}$$

where M_6 is the six-dimensional reduced Planck mass, Λ_6 is the bulk cosmological constant, and $F_{MN} = \partial_M A_N - \partial_N A_M$ is the field strength associated with the $U(1)$ gauge field A_M.

The bulk solution considered as the background in this study is

$$ds_6^2 = r^2 \eta_{\mu\nu} dx^\mu dx^\nu + \frac{dr^2}{f(r)} + f(r) d\phi^2, \tag{2.2}$$

$$A_M dx^M = A(r) d\phi, \tag{2.3}$$

where

$$f(r) = -\frac{\Lambda_6}{10} r^2 - \frac{\mu_b}{r^3} - \frac{b^2}{12 r^6}, \tag{2.4}$$

$$A(r) = \frac{b}{3r^3}. \tag{2.5}$$

2

This solution corresponds to the $\Lambda_{4\pm} \to 0$ limit of the more general solution found in [12], where $\Lambda_{4\pm}$ is the four-dimensional cosmological constant on the branes. (See Appendix A.2. of [12] for this limit.) As explained in [12], this solution is related by a double Wick rotation to a topological black hole with the horizon topology $\mathbf{R}^4$.

Next, we consider embedding of one or two 3-brane sources. As is well known, an object with codimension 2 induces a deficit angle around it as

$$\delta_{\pm} = \frac{\sigma_{\pm}}{M_6^4}, \tag{2.6}$$

where $\delta_{\pm}$ is the deficit angle due to the tension $\sigma_{\pm}$ of the brane. We should note that this formula is valid under the axisymmetry if radial stress is much smaller than energy density [27].

Hereafter, we assume that the function $f(r)$ given by Eq.(2.4) have two positive roots $r = r_{\pm}$ $(0 < r_- < r_+)$ for $f(r) = 0$ and is positive between them $(r_- < r < r_+)$. This requires that the six-dimensional cosmological constant Λ_6 be positive, and thus we assume $\Lambda_6 > 0$ throughout the paper. Since f vanishes at $r_{\pm}$, $r = r_{\pm}$ defines surfaces of codimension 2. Thus we can put the 3-branes at $r_{\pm}$. The periods of the angular coordinate $\Delta\phi$ calculated at $r_{\pm}$ must coincide, thus we get

$$\Delta\phi = \frac{2\pi - \delta_+}{\kappa_+} = \frac{2\pi - \delta_-}{\kappa_-}, \tag{2.7}$$

where $\delta_{\pm}$ are the deficit angles,

$$\kappa_{\pm} \equiv \mp \frac{1}{2} f'(r_{\pm}), \tag{2.8}$$

and a prime denotes derivative with respect to r. This is rewritten as

$$\frac{2\pi - \delta_+}{2\pi - \delta_-} = \frac{\kappa_+}{\kappa_-}, \tag{2.9}$$

which can be regarded as a boundary condition since the l.h.s is specified by the brane sources and the r.h.s. can be written in terms of the bulk parameters μ_b and b.

The background geometry can be specified by the three parameters, the tensions $\sigma_{\pm}$ of the brane at $r_{\pm}$ and the magnetic flux Φ in the bulk. The magnetic flux Φ is given by the equation

$$\frac{\Phi}{\sqrt{(2\pi - \delta_+)(2\pi - \delta_-)}} = -\frac{b}{3\sqrt{\kappa_+\kappa_-}L}\left(\frac{1}{r_-^3} - \frac{1}{r_+^3}\right), \tag{2.10}$$

where $L \equiv \sqrt{10/\Lambda_6}$ and the r.h.s is written only by μ_b and b. If we specify $\sigma_{\pm}$ and Φ, the left hand sides of Eq.(2.9) and Eq.(2.10) are determined. On the other hand,

the right hand sides of these equations are written by μ_b and b. Thus we can solve (2.9) and (2.10) w.r.t. μ_b and b. We can also determine $\Delta\phi$ by Eq.(2.7).

Before going into details, we make some general remarks on the background solution. Our exact braneworld solution captures some essential features of the warped flux compactification, including warped geometry, compactification, moduli stabilization, flux and branes. However, since the higher-dimensional cosmological constant Λ_6 is assumed to be positive, we have to confess that this model of warped flux compactification does not completely mimic the KKLT construction, where the 10-dimensional cosmological constant is not positive but zero. Nonetheless, our model is useful in the sense that it provides a testing ground on which brane gravity with warped flux compactification can be analyzed from higher dimensional viewpoints. Note that many cosmological considerations in the KKLT setup are based on the implicit assumption that 4-dimensional Einstein gravity should be recovered at low energy and that it is very important to see whether this assumption is viable or not from higher dimensional viewpoint. As discussed in [12], the recovery of the 4-dimensional Einstein gravity in warped flux compactification is not as simple as would be expected from the 4-dimensional effective theory. The 4-dimensional gravity is recovered in a rather subtle way as a consequence of bulk dynamics, and the exact solutions in our model were useful to see how this picture works explicitly.

Finally, we explain the limit $\alpha \equiv r_-/r_+ \to 1$ of the bulk geometry. In this limit, the coordinate distance $r_+ - r_-$ between the two branes vanishes, and thus the bulk geometry appears to collapse. However, the proper distance between $r = r_-$ and r_+ does not vanish and the geometry of extra dimensions remains regular. This can be explicitly shown by a coordinate transformation:

$$
\bar{r} = \frac{2r - (r_+ + r_-)}{r_+ - r_-} \quad (-1 \leq \bar{r} \leq 1), \tag{2.11}
$$

$$
\varphi = \Lambda_6 (r_+ - r_-) \phi \tag{2.12}
$$

Even if we take $\alpha \to 1$, the domain of $\bar{r}$ remains finite while that of r vanishes. With this new coordinate system, the metric of the extra dimensions becomes

$$
\frac{dr^2}{f(r)} + f(r)d\phi^2 = \frac{d\bar{r}^2}{4\bar{f}} + \frac{\bar{f}}{\Lambda_6^2}d\varphi^2, \tag{2.13}
$$

where

$$
\begin{aligned}
\bar{f}(\bar{r}) &= \frac{f(r)}{r_+ r_- (\alpha^{-1/2} - \alpha^{1/2})^2} \\
&= \frac{1}{\beta_-^2}\left[-\frac{\gamma_2}{\gamma_1}\left(\frac{2}{\beta_-\bar{r} + \beta_+}\right)^6 + \frac{\beta_+(\alpha^{-1} + \alpha)(\alpha^{-2} + \alpha^2)}{\gamma_1}\left(\frac{2}{\beta_-\bar{r} + \beta_+}\right)^3 \right.
\end{aligned}
$$

4

$$-\left(\frac{\beta_- \bar{r} + \beta_+}{2}\right)^2\Bigg],$$

(2.14)

and we defined

$$\beta_\pm \equiv \alpha^{-1/2} \pm \alpha^{1/2},$$

(2.15)

$$\gamma_n \equiv \sum_{i=0}^{2n} \alpha^{i-n}.$$

(2.16)

The function $\bar{f}$ includes only the background parameter α, and is defined so as to remain finite in the limit of $\alpha \to 1$. Indeed, $\bar{f}$ in this limit is as simple as

$$\bar{f} = \frac{\Lambda_6}{2}\left(1 - \bar{r}^2\right).$$

(2.17)

Thus, the metric (2.13) becomes that of a round sphere of radius $1/\sqrt{2\Lambda_6}$ in this limit.

As we mentioned, the coordinate $\bar{r}$ runs over the finite interval $[-1, 1]$. On the other hand, the period of the coordinate φ appears to collapse since the coefficient $(r_+ - r_-)$ in the definition (2.12) vanishes in the $\alpha \to 1$ limit. Actually, this is not the case. The "surface gravity" $\kappa_\pm$ defined in Eq.(2.8) is written in terms of $\bar{f}$ as

$$\kappa_\pm = \mp\left(r_+ - r_-\right)\partial_{\bar{r}}\bar{f}\left(\pm 1\right).$$

(2.18)

Therefore, the new coordinate φ has the period

$$\Delta\varphi = \Lambda_6\left(r_+ - r_-\right)\Delta\phi = \mp\frac{\Lambda_6}{\partial_{\bar{r}}\bar{f}\left(\pm 1\right)}\left(2\pi - \delta_\pm\right),$$

(2.19)

which is indeed finite, and becomes $\Delta\varphi = 2\pi - \delta_+ = 2\pi - \delta_-$ in the $\alpha \to 1$ limit. Thus, the geometry of extra dimensions is nothing but a round sphere with a deficit angle $\delta_+ = \delta_-$, i.e. a football-shaped extra-dimensions considered in [20, 21].

As we shall see in the next section, when the new coordinates $(\bar{r}, \varphi)$ are used and the KK mass is properly rescaled, the system of the perturbation equations depends on the background parameters through just one parameter α. Thus, we calculate the mass spectra of KK modes in each type of perturbations while changing α from 1 to 0. The perturbation equations written in terms of $(\bar{r}, \varphi)$ remain regular in the $\alpha \to 1$ limit.

3 Dynamical stability

In this section, we consider linear perturbations around the background spacetime described in the previous section in order to show the dynamical stability of this

spacetime. For simplicity we assume that the perturbations are axisymmetric in the extra dimensions. Since the background geometry has the 4-dimensional Poincare symmetry, it is convenient to expand perturbations by scalar-, vector- and tensor-type harmonics of the 4-dimensional Minkowski spacetime. The perturbations are labeled by its type and values of mass squared $m^2 \equiv -\eta^{\mu\nu} k_\mu k_\nu$, where k_μ is the 4D projected wave number of each harmonics.

To study the stability, we take advantages of the fact that for any perturbation type, the perturbation equations are reduced to eigenvalue problems with eigenvalue m^2. We regard the background spacetime to be dynamically stable if the spectrum of m^2 is real and non-negative. In each sector, our strategy to tackle this problem consists of the following four steps:

(i) We show the reality of m^2.

(ii) We rewrite the systems of the perturbation equations into a form which depends on background parameters through just one parameter α. The parameter α runs over the finite interval $[0, 1]$.

(iii) In the $\alpha \to 1$ limit we analytically solve the perturbation equations and show that the spectrum of m^2 is non-negative.

(iv) We numerically evaluate how each eigenvalue m^2 changes as α changes from 1 to 0, and show that the spectrum remains non-negative throughout.

The steps (i) and (ii) make it possible for us to analyze the stability in the α-m^2 plane, i.e. a 2-dimensional space spanned by the paramter α and the eigenvalue m^2. In particular, each eigenvalue generates a curve in the α-m^2 plane as α runs over the interval $[0, 1]$, and what we have to show is that all such curves are in the stable region $m^2 \geq 0$. Thus, provided that each eigenvalue is a continuous function of α, the step (iii) implies that, if there exists a value of α ($0 \leq \alpha \leq 1$) for which the background is unstable, the curve of the lowest eigenvalue must cross the α axis at least once. Finally, the step (iv) proves the stability of the system. In this way, we shall show that our six-dimensional brane world model is dynamically stable against tensor-, vector- and scalar-type linear perturbations. While we can easily show the stability against tensor-type perturbations even without solving the perturbed Einstein equation, we need to perform numerical calculations in the step (iv) for scalar- and vector-type perturbations. For completeness, we shall perform numerical calculations for tensor perturbations as well and show first few KK modes.

In the analysis of each sector, we shall require regularity of physically relevant, geometrical quantities such as the Ricci scalar of the induced metric on the brane,

the tetrad components of the bulk Weyl tensor evaluated on the brane, etc. This is because we are adopting the thin brane approximation, i.e. (2.6), and all we can and should trust is what is obtained within the validity of this approximation. If e.g. the Ricci scalar of the induced metric were singular then our approximation would be invalidated. It is of course possible to regularize the singularity by introducing a finite thickness of the brane. However, in this case the natural cutoff of the low energy effective theory is the inverse of the thickness, and in general the regularized "would-be singularity" is not expected to be below the cutoff scale. This simply means that we need a UV completion, e.g. the microphysical description of the brane, to describe the physics of the regularized "would-be singularity". Therefore, in general we have two options: (a) to specify a fundamental theory such as string theory as a UV completion and go on; or (b) to concentrate on modes which are within the validity of the effective theory. In this paper we adopt the latter attitude, assuming the existence of a good UV completion but never using its properties. This is the reason why we adopt the thin brane approximation and require the regularities.

Let us now start the analysis. To begin with, we expand the perturbed metric by harmonics of the 4-dimensional Minkowski spacetime as

$$\delta g_{MN} dx^M dx^N = h_{rr} Y dr^2 + 2h_{r\phi} Y dr d\phi + h_{\phi\phi} Y d\phi^2$$
$$+ 2\Big(h_{(T)r} V_{(T)\mu} + h_{(L)r} V_{(L)\mu}\Big) dr dx^\mu + 2\Big(h_{(T)\phi} V_{(T)\mu} + h_{(L)\phi} V_{(L)\mu}\Big) d\phi dx^\mu$$
$$+ \Big(h_{(T)} T_{(T)\mu\nu} + h_{(LT)} T_{(LT)\mu\nu} + h_{(LL)} T_{(LL)\mu\nu} + h_{(Y)} T_{(Y)\mu\nu}\Big) dx^\mu dx^\nu, \qquad (3.1)$$

where Y, $V_{(T,L)}$ and $T_{(T,LT,LL,Y)}$ are scalar, vector and tensor harmonics, respectively. See Appendix A.1 for definitions of the harmonics. The coefficients h_{rr}, $h_{r\phi}$, $h_{\phi\phi}$, $h_{(T,L)r}$, $h_{(T,L)\phi}$ and $h_{(T,LT,LL,Y)}$ are supposed to depend only on r. In the same manner, the perturbations of the $U(1)$ gauge field can be expanded as

$$\delta A_M dx^M = a_r Y dr + a_\phi Y d\phi + \Big(a_{(T)} V_{(T)\mu} + a_{(L)} V_{(L)\mu}\Big) dx^\mu. \qquad (3.2)$$

Here the coefficient a_r, a_ϕ and $a_{(T,L)}$ are also supposed to depend only on r due to axisymmetry.

The Einstein equations and the Maxwell equations are decomposed into three groups, each of which contains only variables belonging to one of the following three sets of variables $\{h_{(T)}\}$, $\{h_{(T)r}, h_{(T)\phi}, h_{(LT)}, a_{(T)}\}$ and $\{h_{rr}, h_{r\phi}, h_{\phi\phi}, h_{(L)r}, h_{(L)\phi}, h_{(LL,Y)}, a_r, a_\phi, a_{(L)}\}$. Variables belonging to each set are called tensor type, vector type and scalar type, respectively. It should be noted that these variables include degrees of freedom of gauge transformation, which are explicitly given in Appendix A.2.

7

3.1 Tensor-type perturbation

Our first task is to show that $m^2 \geq 0$ for any non-vanishing tensor perturbations. We first derive the evolution equation for tensor perturbations in subsection 3.1.1. From this equation and regularity of h and h', we can show $m^2 \geq 0$ without solving the evolution equation. We also show in subsection 3.1.1 that the system of the evolution equation and the boundary condition can be rewritten in the form which depends on the background parameters through just one parameter $\alpha \equiv r_-/r_+$. Then, we seek the analytic solution for $\alpha = 1$ in subsection 3.1.2. Using this result, we numerically solve the perturbation equations by relaxation method in subsection 3.1.3. Detailed explanation of relaxation method is given in Appendix A.3.

3.1.1 Basic equations

For tensor perturbations,

$$
\begin{aligned}
ds_6^2 &= r^2(\eta_{\mu\nu} + hY s_{\mu\nu})dx^\mu dx^\nu + \frac{dr^2}{f} + fd\phi^2, \\
A_M dx^M &= Ad\phi,
\end{aligned}
\tag{3.3}
$$

where the perturbation is specified by the function h of r and the harmonics $Y \equiv \exp(ik_\mu x^\mu)$. The symmetric polarization tensor $s_{\mu\nu}$ satisfies $s^\mu_\mu = k^\mu s_{\mu\nu} = 0$ for $k^\mu k_\mu \neq 0$, or $s^\mu_\mu = k^\mu s_{\mu\nu} = \tau^\mu s_{\mu\nu} = 0$ for $k^\mu k_\mu = 0$, where τ^μ is a constant timelike vector. There is no relevant equation coming from the Maxwell equation, and the perturbed Einstein equation becomes

$$
\frac{1}{r^2}\left(r^4 fh'\right)' + m^2 h = 0,
\tag{3.4}
$$

where a prime denotes derivative with respect to r and $m^2 \equiv -\eta^{\mu\nu}k_\mu k_\nu$. The regularity of the four-dimensional Ricci scalar of the induced metric on the brane requires that h should be regular. With the above equation for h, the regularity of the tetrad components of the six-dimensional Weyl tensor on the brane requires that h' should also be regular. Indeed, provided that h is regular, $C^r{}_{\mu r \nu}/r$ is regular at $r = r_\pm$ if and only if h' is regular.

With the regularity of h and h' at $r = r_\pm$, it is easy to show that $m^2 \geq 0$ for any non-vanishing solutions:

$$
m^2 \int_{r_-}^{r_+} dr\ r^2 h^2 = -\int_{r_-}^{r_+} dr\ h\left(r^4 fh'\right)' = \int_{r_-}^{r_+} dr\ (r^2 h')^2 f \geq 0,
\tag{3.5}
$$

where we have done integration by parts and used the fact that $f(r_\pm) = 0$. The equality holds if and only if $h' = 0$ in the region $r_- \leq r \leq r_+$. Thus, there is no instability in the tensor sector and the zero mode is $h = const$.

The above differential equation (3.4) include the two background parameters (μ_b, b), or equivalently (r_+, r_-). From now, we show this equation can be rewritten into a form which includes only one parameter $\alpha = r_-/r_+$. First of all, we perform a coordinate transformation Eq.(2.11). In terms of $\bar{r}$, the Eq.(3.4) becomes

$$\frac{1}{(\beta_-\bar{r} + \beta_+)^2}\partial_{\bar{r}}\left((\beta_-\bar{r} + \beta_+)^4\,\bar{f}\partial_{\bar{r}}h\right) + \tilde{m}^2 h = 0, \tag{3.6}$$

where we defined

$$\tilde{m}^2 = \frac{m^2}{r_+ r_-}. \tag{3.7}$$

The function $\bar{f}$ and thus the Eq.(3.6) include only the parameter α. The boundary conditions for h are obtained by assuming that h can be expanded in the Taylor series at $r = r_\pm$:

$$\partial_{\bar{r}}h + \left.\frac{\tilde{m}^2}{(\beta_-\bar{r} + \beta_+)^2\,\partial_{\bar{r}}\bar{f}}h\right|_{\bar{r}\to\pm1} = 0. \tag{3.8}$$

Thus, it is sufficient that we solve the perturbation equation with a variety of α.

3.1.2 Analytic solution for $\alpha = 1$

Here, we give the solution of h for $\alpha = 1$, which is used when we numerically solve the perturbation for general α in the next subsection. Substituting $\bar{f}$ into Eq.(3.6) and taking $\alpha \to 1$, we get

$$\partial_{\bar{r}}\left(\left(1 - \bar{r}^2\right)\partial_{\bar{r}}h\right) + \mu^2 h = 0, \tag{3.9}$$

where

$$\mu^2 \equiv \frac{\tilde{m}^2}{2\Lambda_6}. \tag{3.10}$$

This can be solved analytically as

$$h = CP_\nu\left(\bar{r}\right) + DQ_\nu\left(\bar{r}\right), \tag{3.11}$$

where C and D are normalization constants. P_ν and Q_ν are the Legendre functions of the first and second kind, respectively. ν is related to μ^2 as $\mu^2 = \nu(\nu + 1)$. The

regularity of h at $\bar{r} = 1$ requires $D = 0$, and that at $\bar{r} = -1$ does P_ν not to diverge there. This is realized if only if ν is non-negative integer. The mass spectrum is then determined as

$$\mu^2 = \nu(\nu + 1) \quad (\nu = 0, 1, 2, \cdots). \tag{3.12}$$

The zero mode (a mode with $m^2 = 0$) is $h = const.$ and all other modes (i.e. Kaluza-Klein modes) have positive m^2. Thus, the background spacetime for $\alpha = 1$ is dynamically stable against tensor perturbations.

3.1.3 Numerical solution of KK modes

Here we obtain the first few KK modes of tensor-type perturbation by numerically solving the perturbed Einstein equation (3.6) with the boundary condition (3.8). For this purpose, the relaxation method is useful. Detailed explanation of the relaxation method is given in Appendix A.3. Here we give an outline of this method. We first rewrite the second order differential equation (3.6) to a system of two first order differential equations by defining $\partial_{\bar{r}} h$ as well as h as a dependent variable. Then, the differential equations are replaced by finite-difference equations on a mesh of M points that covers the range of the integration. We start with an arbitrary trial solution which does not necessarily satisfy the desired finite-difference equations, nor the required boundary conditions. The successive iteration, now called relaxation, will adjust all the values on the mesh so as to realize a closer agreement with finite-difference equations and, simultaneously, with the boundary conditions. Good initial guesses are the key of efficiency in the relaxation method. Here we have to solve the problem many times, each time with a slightly different value of α. In this case, the previous solution will be a good initial guess when α is changed, and it will make relaxation work well. As shown in the previous subsection, the perturbation equations can be analytically solved for $\alpha = 1$. Thus, we solve the problem while changing α from 1 to 0.

We show the first four KK mode solutions of h for a given value of α in figure 1. When we plot the solutions, normalization is determined by using the generalized Klein-Gordon norm. See Appendix A.4 for its derivation. For tensor perturbations, it is defined by

$$(\Phi, \Psi)_{KG} \equiv -i\frac{M_6^4 \Delta\phi}{8} \int d^3\mathbf{x} \int dr r^2 \eta^{\mu\mu'} \eta^{\nu\nu'} \left(\Phi_{\mu\nu} \partial_t \Psi^*_{\mu'\nu'} - \Psi^*_{\mu\nu} \partial_t \Phi_{\mu'\nu'}\right). \tag{3.13}$$

Substituting $\Phi_{\mu\nu} = \int d^4k \, h_k s_{\mu\nu} Y$ and $\Psi_{\mu'\nu'} = \int d^4k' \, h_{k'} s_{\mu'\nu'} Y$, we get

$$(\Phi, \Psi)_{KG} = (k_0 + k'_0) \, \delta^3 (\mathbf{k} - \mathbf{k}') \frac{M_6^4 \Delta\phi}{8} \int dr r^2 h_{n_1}(r) h_{n_2}(r), \tag{3.14}$$

where subscript n_k means h_{n_k} is the solution of tensor perturbations with the eigenvalue $m_{n_k}^2$, and we normalized the symmetric polarization tensor as $s_{\mu\nu}s^{\mu\nu} = 1$. In terms of the coordinate $(\bar{r}, \varphi)$, this becomes

$$
\begin{aligned}
(\Phi, \Psi)_{KG} &= (k_0 + k_0')\,\delta^3\,(\mathbf{k} - \mathbf{k}')\,\Delta\varphi\frac{M_6^4 r_+ r_-}{16\Lambda_6} \int d\bar{r}\frac{(\beta_-\bar{r} + \beta_+)^2}{4}h_{n_1}(\bar{r})h_{n_2}(\bar{r}) \\
&\equiv (k_0 + k_0')\,\delta^3\,(\mathbf{k} - \mathbf{k}')\,\Delta\varphi\frac{M_6^4 r_+ r_-}{16\Lambda_6}\,(h_{n_1}, h_{n_2})\,.
\end{aligned}
\tag{3.15}
$$

We normalize the solution by

$$
\frac{(h_{n_1}, h_{n_2})}{(1, 1)} = \delta_{n_1 n_2},
\tag{3.16}
$$

so that the zero mode solution is normalized as $h(\bar{r}) = 1$. We can easily prove the orthogonality between modes with different m^2 by using the equation of motion for h, (3.6).

Finally, we show the spectrum of the mass squared of first four KK modes as a function of α. The mass squared that has physical meaning is $m_\pm^2 \equiv -r_\pm^{-2}\eta^{\mu\nu}k_\mu k_\nu$, which are the ones observed on the brane at $r = r_\pm$. They are related to $\tilde{m}^2$ as $m_\pm^2 = \alpha^{\pm 1}\tilde{m}^2$. We plot the spectrum of m_+^2 in figure 2. As is easily seen, m_+^2 is non-negative for the entire range of α. Therefore, the background spacetime we consider is dynamically stable in the tensor-type sector.

We also notice that m_+^2 remains finite in $\alpha \to 0$ limit, which implies $m_+^2 \propto \alpha^0$, whereas $m_-^2 \propto \alpha^{-2}$ for $\alpha \to 0$. This result is consistent with our previous study [12], where we analyzed how the Hubble expansion rate $H_\pm$ on each brane changes when the tension of the brane changes. In that paper we also considered higher-order corrections of the effective Friedmann equation with respect to $H_\pm$. The result is that the higher-order corrections appear when $H_\pm$ get larger than critical values $H_{*\pm}$. For $\alpha \to 0$, we found that $H_{*\pm}$ behave as

$$
H_{*+}^2 \propto \alpha^0, \qquad H_{*-}^2 \propto \alpha^{-2}.
\tag{3.17}
$$

Since the higher-order corrections are caused by the KK modes, this energy scale corresponds to their mass measured on each brane. Thus, the behavior of $m_\pm^2$ we obtained in the $\alpha \to 0$ is consistent with our previous result in [12].

3.2 Vector-type perturbation

Next we show that $m^2 > 0$ for any non-vanishing vector perturbations satisfying relevant boundary conditions. We first derive the perturbed Einstein equations in
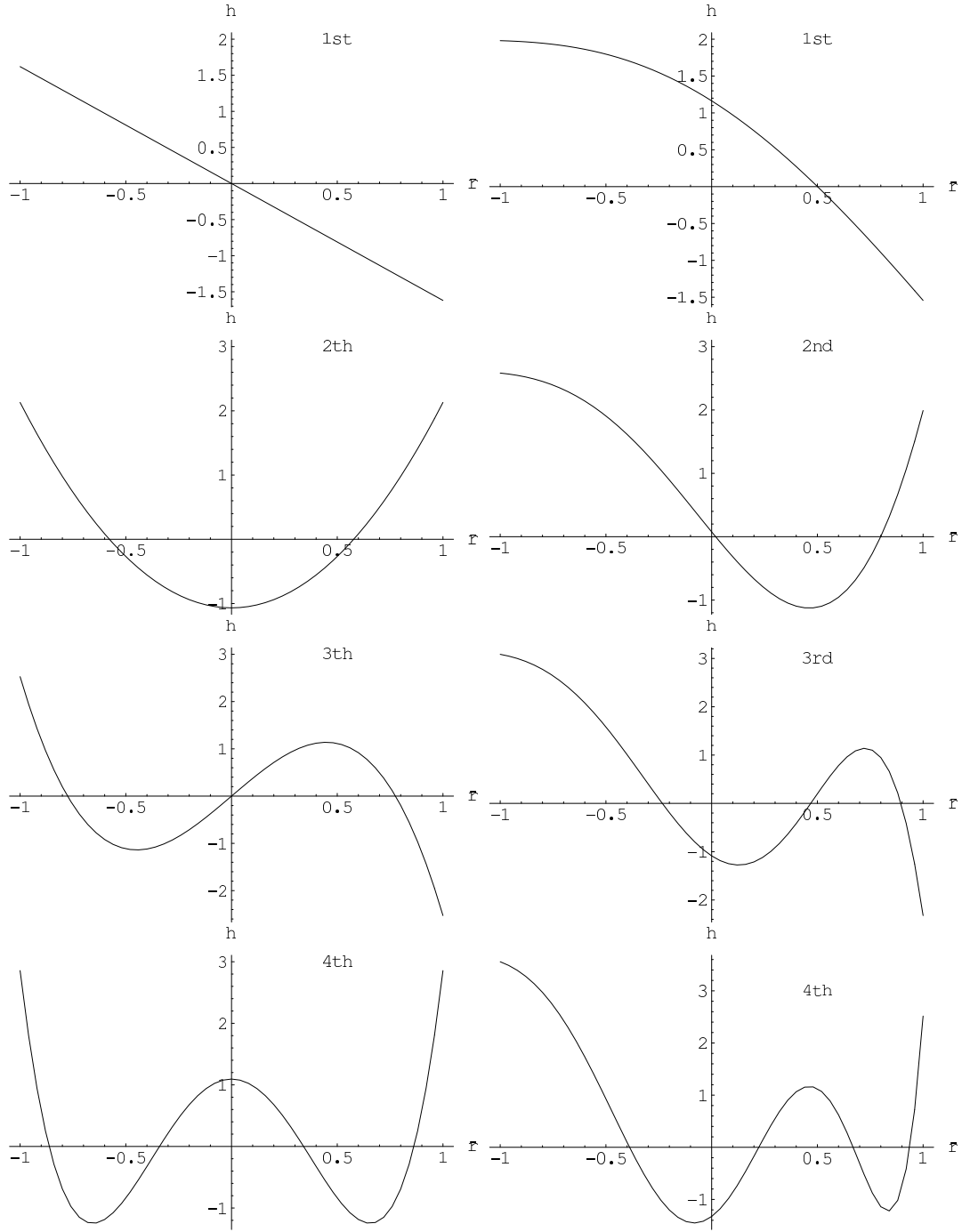
Figure 1: The solution h of the first four KK modes for $\alpha = 1$ (left) and $\alpha = 0.11$ (right). The normalization is determined by using the generalized Klein-Gordon norm (see the text). Number of points of the mesh is taken to be 51.
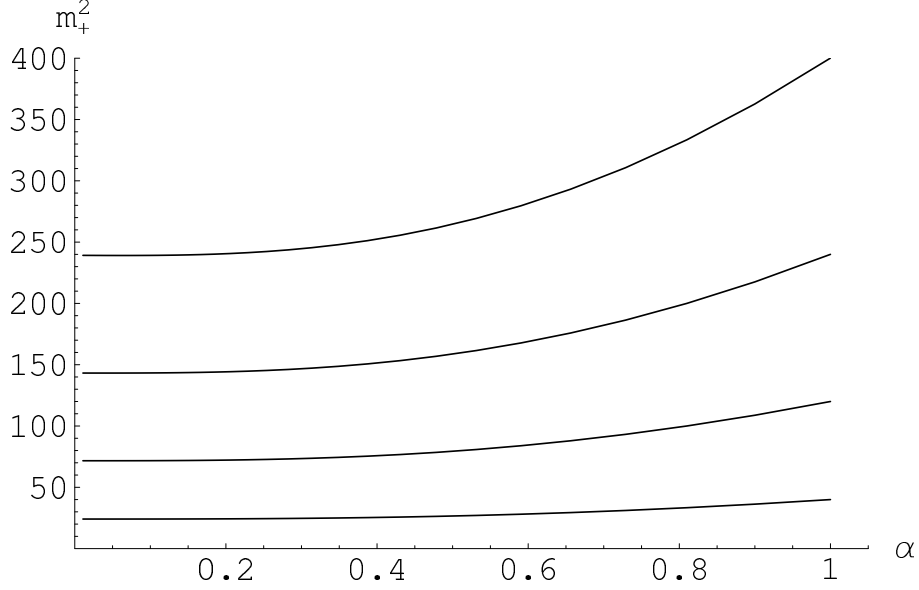
Figure 2: The spectrum of m_+^2 for tensor perturbations as a function of α.

subsection 3.2.1. There are two physical degrees of freedom for vector perturbations. Their differential equations are easily written into a form which is manifestly hermite, therefore m^2 is real. They are also written into a form including only the background parameter α. We then derive the boundary conditions from those equations, by assuming that the two perturbation variables can be Taylor expanded with respect to $\bar{r} \pm 1$. In subsection 3.2.2, we summarize the analytic solution for $\alpha = 1$. Using this result, we numerically solve the perturbation equations by relaxation method in subsection 3.2.3.

3.2.1 Basic equations

For vector perturbations,

$$
\begin{aligned}
ds_6^2 &= r^2 \eta_{\mu\nu} dx^\mu dx^\nu + 2(h_{(T)r} dr + h_{(T)\phi} d\phi) V_{(T)\mu} dx^\mu + \frac{dr^2}{f} + f d\phi^2, \\
A_M dx^M &= a_{(T)} V_{(T)\mu} dx^\mu + A d\phi,
\end{aligned}
\tag{3.18}
$$

where the perturbation is specified by the functions $\{h_{(T)r}, h_{(T)\phi}, a_{(T)}\}$ of r. As for gauge fixing, see Appendix A.2. The (LT)- and $(T)r$-components of the Einstein equation give

$$
h_{(T)r} = \frac{C}{r^2 f},
\tag{3.19}
$$

13

where C is an arbitrary constant for $m^2 = 0$, or $C = 0$ for $m^2 \neq 0$. The (T)-component of the Maxwell equation and the $(T)\phi$-component of the Einstein equation are reduced to

$$
\begin{aligned}
\left(r^2 f \Phi'_{(V)1}\right)' - \sqrt{2} r^4 A' \Phi'_{(V)2} + m^2 \Phi_{(V)1} &= 0, \\
\left(r^6 \Phi'_{(V)2}\right)' + \sqrt{2} r^4 A' \Phi'_{(V)1} + \frac{m^2 r^4}{f} \Phi_{(V)2} &= 0,
\end{aligned}
\tag{3.20}
$$

where

$$
\begin{aligned}
\Phi_{(V)1} &\equiv \sqrt{2} a_{(T)}, \\
\Phi_{(V)2} &\equiv \frac{h_{(T)\phi}}{r^2}.
\end{aligned}
\tag{3.21}
$$

For $m^2 = 0$, if the constant C were nonzero then by going to the gauge $\bar{h}_{(T)r} = 0$ by the gauge transformation (A.25), the metric component $\bar{h}_{(LT)}$ given by (A.22) would diverge on the brane. Thus, the regularity of the induced metric requires that $C = 0$ for $m^2 = 0$. (For $m^2 \neq 0$, it has already been shown that $C = 0$.)

The regularity of the field strength for the pull-back of $A_M dx^M$ on the brane requires that $\Phi_{(V)1}$ is regular at $r = r_\pm$. With the above equations (3.20), the regularity of the tetrad components of the six-dimensional Weyl tensor on the brane requires that $\Phi_{(V)2}/f$, $\Phi'_{(V)2}$ and $\sqrt{f}\Phi'_{(V)1}$ should be finite at $r = r_\pm$. These regularity conditions are enough to make equations (3.20) hermite. Therefore, m^2 is real.

As in the case of tensor perturbations, we need to rewrite the above equations (3.20) into a form which includes only α and which makes it possible for us to take the $\alpha \to 1$ limit without any divergence. Actually, while the variables $\Phi_{(V)1}$ and $\Phi_{(V)2}$ were useful to show the reality of m^2, for numerical calculation there is a more convenient choice of variables. The $(T)\varphi$-component of the metric perturbation $h_{(T)\varphi}$ in the coordinate $(\bar{r}, \varphi)$ is related to $h_{(T)\phi}$ as

$$
h_{(T)\phi} = \Lambda_6 \sqrt{r_+ r_-} \beta_- h_{(T)\varphi}.
\tag{3.22}
$$

The coefficient β_- in the above equation vanishes in the $\alpha \to 1$ limit. Thus, we rescale $h_{(T)\phi}$ as

$$
\tilde{h}_{(T)\phi} = \frac{1}{\sqrt{\Lambda_6 r_+ r_-} \beta_-} h_{(T)\phi},
\tag{3.23}
$$

which approaches to $\sqrt{\Lambda_6} h_{(T)\varphi}$ in the $\alpha \to 1$ limit. Using this variable, we can rewrite the above equations (3.20) as

$$
\partial_{\bar{r}}^2 a_{(T)} + \left(\frac{\partial_{\bar{r}} \bar{f}}{\bar{f}} + \frac{2\beta_-}{\beta_- \bar{r} + \beta_+}\right) \partial_{\bar{r}} a_{(T)} + \frac{8\sqrt{2}\Lambda_6}{\bar{f}\left(\beta_- \bar{r} + \beta_+\right)^4} \sqrt{\frac{3}{\gamma_1}} \sqrt{\frac{\gamma_2}{5}} \left(\partial_{\bar{r}} \tilde{h}_{(T)\phi}\right)
$$

14

$$-\frac{2\beta_-}{\beta_-\bar{r}+\beta_+}\tilde{h}_{(T)\phi}\Bigg) + \frac{\tilde{m}^2}{\bar{f}\left(\beta_-\bar{r}+\beta_+\right)^2}a_{(T)} = 0$$

$$\partial_{\bar{r}}^2\tilde{h}_{(T)\phi} + \frac{2\beta_-}{\beta_-\bar{r}+\beta_+}\partial_{\bar{r}}\tilde{h}_{(T)\phi} - \frac{16\sqrt{2}}{\left(\beta_-\bar{r}+\beta_+\right)^4}\sqrt{\frac{3}{\gamma_1}}\sqrt{\frac{\gamma_2}{5}}\partial_{\bar{r}}a_{(T)}$$

$$-6\left(\frac{\beta_-}{\beta_-\bar{r}+\beta_+}\right)^2\tilde{h}_{(T)\phi} + \frac{\tilde{m}^2}{\bar{f}\left(\beta_-\bar{r}+\beta_+\right)^2}\tilde{h}_{(T)\phi} = 0, \tag{3.24}$$

where $\gamma_n = \sum_{i=0}^{2n}\alpha^{i-n}$ as we defined above. The boundary conditions are obtained by assuming that $\Phi_{(V)1}$ and $\tilde{\Phi}_{(V)2}$ can be expanded in the Taylor series at $r = r_\pm$:

$$\tilde{h}_{(T)\phi}\Big|_{\bar{r}\to\pm1} = 0$$

$$\partial_{\bar{r}}a_{(T)} + \frac{8\sqrt{2}\Lambda_6}{\left(\beta_-\bar{r}+\beta_+\right)^4\partial_{\bar{r}}\bar{f}}\sqrt{\frac{3}{\gamma_1}}\sqrt{\frac{\gamma_2}{5}}\left(\partial_{\bar{r}}\tilde{h}_{(T)\phi} - \frac{2\beta_-}{\beta_-\bar{r}+\beta_+}\tilde{h}_{(T)\phi}\right)$$

$$+\frac{\tilde{m}^2}{\left(\beta_-\bar{r}+\beta_+\right)^2\partial_{\bar{r}}\bar{f}}a_{(T)}\Big|_{\bar{r}\to\pm1} = 0. \tag{3.25}$$

3.2.2 Analytic solution for $\alpha = 1$

The system of the perturbation equations (3.24) becomes simple in the $\alpha \to 1$ limit, and then can be solved analytically. Here we summarize those solutions, which are used when we numerically solve the equations (3.24) and (3.25) for general α. Taking $\alpha \to 1$ and using the equation (2.17), the equations (3.24) become

$$\partial_{\bar{r}}\left[(1-\bar{r}^2)\partial_{\bar{r}}a_{(T)}\right] + \sqrt{2\Lambda_6}\partial_{\bar{r}}h_{(T)\varphi} + \mu^2 a_{(T)} = 0,$$

$$\partial_{\bar{r}}^2 h_{(T)\varphi} - \sqrt{\frac{2}{\Lambda_6}}\partial_{\bar{r}}a_{(T)} + \frac{\mu^2}{1-\bar{r}^2}h_{(T)\varphi} = 0, \tag{3.26}$$

where

$$\mu^2 \equiv \frac{\tilde{m}^2}{2\Lambda_6}. \tag{3.27}$$

The absence of the zero mode solution can be easily shown using the above differential equations and the regularity of the variables at the boundaries [28]. In the following, we consider the case of $\mu^2 \neq 0$.

By the change of variables

$$\Psi_1 \equiv a_{(T)},$$

$$\Psi_2 \equiv -\sqrt{2\Lambda_6}\partial_{\bar{r}}h_{(T)\varphi} + 2a_{(T)}, \tag{3.28}$$

15

the above equations become

$$\partial_{\bar{r}} \left[(1 - \bar{r}^2) \partial_{\bar{r}} \Psi_1 \right] - (\Psi_2 - 2\Psi_1) + \mu^2 \Psi_1 = 0,$$
$$\partial_{\bar{r}} \left[(1 - \bar{r}^2) \partial_{\bar{r}} \Psi_2 \right] + \mu^2 (\Psi_2 - 2\Psi_1) = 0. \tag{3.29}$$

The original variables are written in terms of Ψ_1 and Ψ_2 as

$$a_{(T)} = \Psi_1 = \frac{1}{2\mu^2} \left\{ \partial_{\bar{r}} \left[(1 - \bar{r}^2) \partial_{\bar{r}} \Psi_2 \right] + \mu^2 \Psi_2 \right\},$$
$$h_{(T)\varphi} = \frac{1 - \bar{r}^2}{\sqrt{2\Lambda_6}\mu^2} \partial_{\bar{r}} \Psi_2. \tag{3.30}$$

This set of equations can be rewritten as

$$\partial_{\bar{r}} \left[(1 - \bar{r}^2) \partial_{\bar{r}} E_{\pm} \right] + \lambda_{\pm} E_{\pm} = 0, \tag{3.31}$$

where

$$E_{\pm} \equiv \partial_{\bar{r}} \left[(1 - \bar{r}^2) \partial_{\bar{r}} \Psi_2 \right] + \lambda_{\mp} \Psi_2,$$
$$\lambda_{\pm} \equiv \mu^2 + 1 \pm \sqrt{2\mu^2 + 1}. \tag{3.32}$$

Thus, the general solution is

$$\Psi_2 = C_+ P_{\nu_+}(\bar{r}) + C_- P_{\nu_-}(\bar{r}) + D_+ Q_{\nu_+}(\bar{r}) + D_- Q_{\nu_-}(\bar{r}), \tag{3.33}$$

where $C_{\pm}$ and $D_{\pm}$ are constants and $\nu_{\pm}$ is a solution to

$$\nu_{\pm}(\nu_{\pm} + 1) = \lambda_{\pm}. \tag{3.34}$$

From the regularity of $\Phi_{(V)1}$, $\sqrt{f}\Phi'_{(V)1}$, $\Phi_{(V)2}/f$ and $\Phi'_{(V)2}$, we can show that Ψ_1, $\sqrt{1 - \bar{r}^2}\partial_{\bar{r}}\Psi_1$, Ψ_2 and $\partial_{\bar{r}}\Psi_2$ should be finite at the boundaries. At $\bar{r} = 1$, Ψ_2 is regular only if $D_+ + D_- = 0$ whereas the regularity of Ψ_1 is reduced through the differential equations to $(\lambda_+ - \mu^2) D_+ + (\lambda_- - \mu^2) D_- = 0$. Since $\lambda_+ \neq \lambda_-$ for $\mu^2 \neq -1/2$, we obtain $D_+ = D_- = 0$. At $\bar{r} = -1$, the regularity conditions for the variables become

$$C_+ \sin\nu_+\pi + C_- \sin\nu_-\pi = 0,$$
$$C_+\lambda_+ \sin\nu_+\pi + C_-\lambda_- \sin\nu_-\pi = 0. \tag{3.35}$$

Non-trivial solutions can exist if $\nu_+ \in \mathbb{Z}$ or $\nu_- \in \mathbb{Z}$. In general we can choose non-negative $\nu_{\pm}$'s so that these conditions are explicitly written down as

$$\nu_{\pm} = \frac{-1 + \sqrt{4\mu^2 + 5 \pm 4\sqrt{2\mu^2 + 1}}}{2} = 0, 1, 2, \ldots. \tag{3.36}$$

Therefore the KK mass spectrum for vector perturbation is obtained. The case of $\mu^2 \neq -1/2$ can be excluded [28].

16

3.2.3 Numerical solution of KK modes

Here we obtain the first few KK modes of vector type perturbations by numerically solving the perturbed Einstein equation (3.24) and the junction condition (3.25). We rewrite the system of two second order differential equations (3.24) to a system of four first order differential equations by defining $\partial_{\bar{r}} a_{(T)}$ and $\partial_{\bar{r}} \tilde{h}_{(T)\phi}$ as well as $a_{(T)}$ and $\tilde{h}_{(T)\phi}$ as dependent variables. Here, we take the number of points on a mesh $M = 101$. We solve the problem while changing α from 1 to 0, each time with a slightly different value of α. Starging from $\alpha = 1$, the analytic solutions for $\alpha = 1$ presented in the previous subsection are used as a trial solution.

Figure 3 and 4 show the first four KK mode solutions of $a_{(T)}$ and $\tilde{h}_{(T)\phi}$ for $\alpha = 1.0$ and 0.31. The normalization is determined by using the generalized Klein-Gordon norm as in tensor perturbations. For vector perturbations, it is defined by

$$(\tilde{\Phi}, \tilde{\Psi})_{KG} \equiv -i \frac{M_6^4 \Delta\phi}{2} \int d^3\mathbf{x} \int dr \eta^{\mu\nu} \left[\left(\tilde{\Phi}_{1\mu} \partial_t \tilde{\Psi}_{1\nu}^* - \tilde{\Psi}_{1\mu}^* \partial_t \tilde{\Phi}_{1\nu} \right) + \frac{r^4}{f} \left(\tilde{\Phi}_{2\mu} \partial_t \tilde{\Psi}_{2\nu}^* - \tilde{\Psi}_{2\mu}^* \partial_t \tilde{\Phi}_{2\nu} \right) \right]. \tag{3.37}$$

See the Appendix A.4 for the derivation. In terms of $a_{(T)}$ and $h_{(T)\phi}$, this is written as

$$(\tilde{\Phi}, \tilde{\Psi})_{KG} = (k_0 + k_0') \delta^3 (\mathbf{k} - \mathbf{k}') \frac{M_6^4 \Delta\phi}{2} \int dr \left[2 a_{(T)}^{(n_1)} a_{(T)}^{(n_2)} + \frac{1}{f} h_{(T)\phi}^{(n_1)} h_{(T)\phi}^{(n_2)} \right], \tag{3.38}$$

where superscript n_k means that $a_{(T)}^{(n_k)}$ and $h_{(T)\phi}^{(n_k)}$ are the solutions of vector perturbations with eigenvalue $m_{n_k}^2$, and we normalized the constant vector as $u_\mu u^\mu = 1$. Using the coordinate $(\bar{r}, \varphi)$ and the rescaled variable $\tilde{h}_{(T)\phi}^{(n_k)}$, the Klein-Gordon norm is further rewritten as

$$
\begin{aligned}
(\tilde{\Phi}, \tilde{\Psi})_{KG} &= (k_0 + k_0') \delta^3 (\mathbf{k} - \mathbf{k}') \frac{M_6^4 \Delta\varphi}{4\Lambda_6} \int d\bar{r} \left[2 a_{(T)}^{(n_1)} a_{(T)}^{(n_2)} + \frac{\Lambda_6}{\tilde{f}} \tilde{h}_{(T)\phi}^{(n_1)} \tilde{h}_{(T)\phi}^{(n_2)} \right] \\
&\equiv (k_0 + k_0') \delta^3 (\mathbf{k} - \mathbf{k}') \frac{M_6^4 \Delta\varphi}{4\Lambda_6} \left(a_{(T)}^{(n_1)}, \tilde{h}_{(T)\phi}^{(n_1)} \middle| a_{(T)}^{(n_2)}, \tilde{h}_{(T)\phi}^{(n_2)} \right).
\end{aligned}
\tag{3.39}
$$

We normalize the solution by

$$\left(a_{(T)}^{(n_1)}, \tilde{h}_{(T)\phi}^{(n_1)} \middle| a_{(T)}^{(n_2)}, \tilde{h}_{(T)\phi}^{(n_2)} \right) = \delta_{n_1 n_2}. \tag{3.40}$$

We can easily prove the orthogonality between modes with different m^2 by using the equation of motion for vector perturbations.

Finally, we show the spectrum of m_+^2 for the first four KK modes as a function of α in figure 5. We find that m_+^2 is non-negative for the entire range of α. Therefore, the background spacetime is also dynamically stable in the vector-type sector.
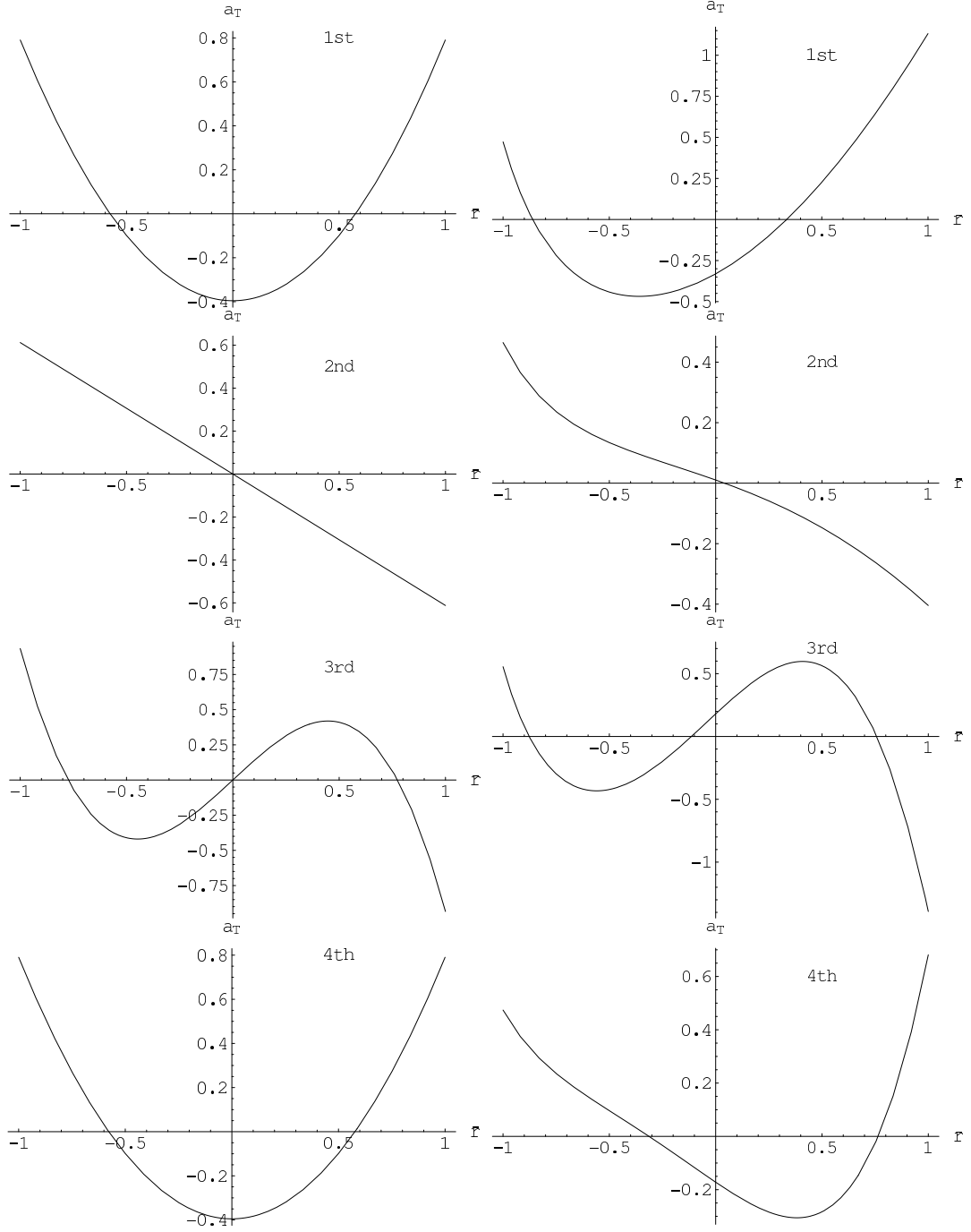
17

Figure 3: The solution $a_{(T)}$ of the first four KK modes for $\alpha = 1$ (left) and $\alpha = 0.31$ (right). The normalization is determined by using the generalized Klein-Gordon norm (see the text). Number of points of the mesh is taken to be 101.
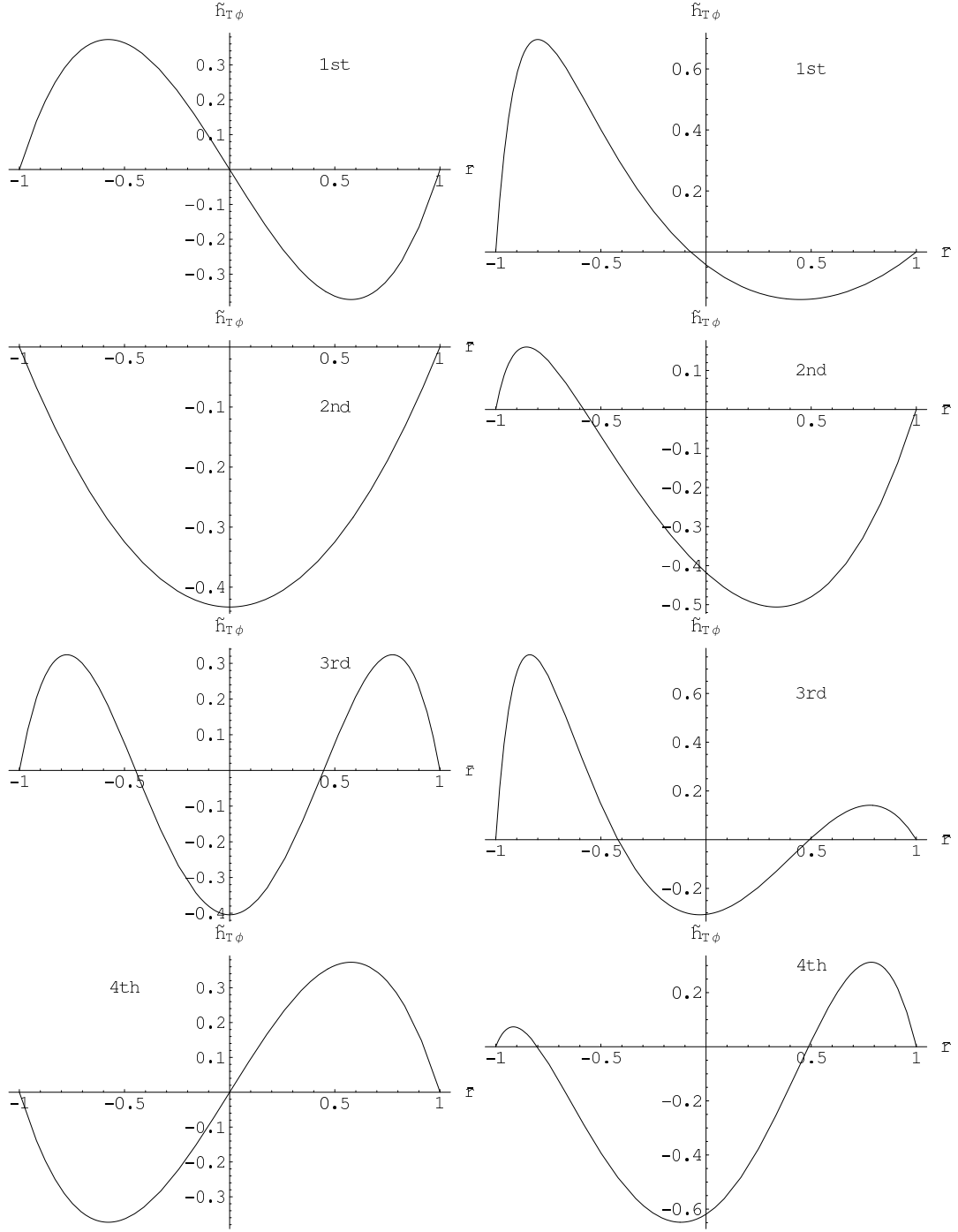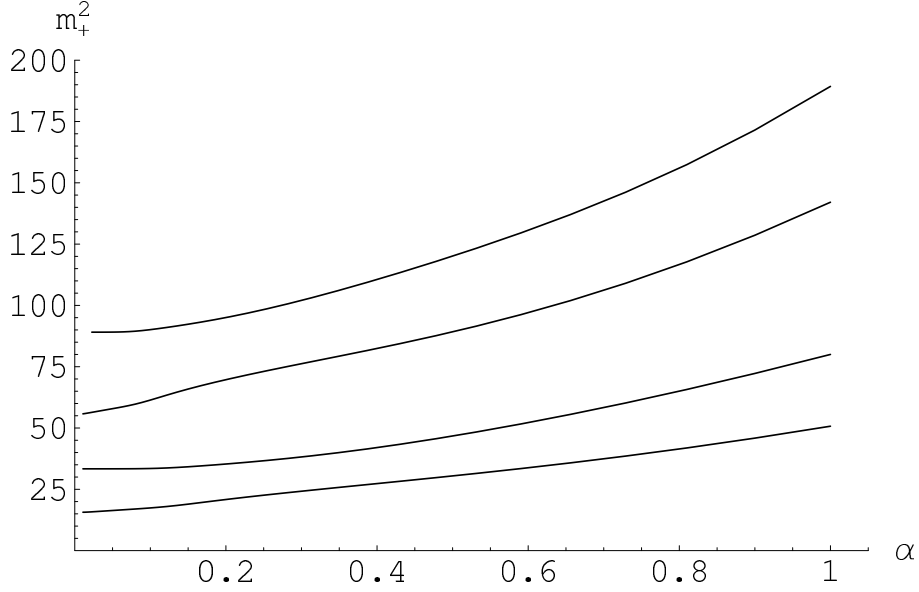
Figure 4: The solution $\tilde{h}_{(T)\phi}$ of the first four KK modes for $\alpha = 1$ (left) and $\alpha = 0.31$ (right). The normalization is determined by using the generalized Klein-Gordon norm (see the text). Number of points of the mesh is taken to be 101.

Figure 5: The spectrum of m_+^2 for vector perturbations as a function of α.

The behavior of m_+^2 in $\alpha \to 0$ limit has the same feature as in tensor perturbations. It remains finite in this limit, that is, $m_+^2 \propto \alpha^0$. As was mentioned in subsection 3.1.3, this result is consistent with our previous study [12], where we have found that the energy scale H_{*+} at which the KK modes begin to modify the effective Friedmann equation on the brane at $r = r_+$ behaves as $H_{*+}^2 \propto \alpha^0$ when $\alpha \sim 0$.

3.3 Scalar-type perturbation

Finally, we show that $m^2 > 0$ for any non-vanishing scalar perturbations satisfying relevant boundary conditions. We first derive the perturbed Einstein equations and the boundary conditions in subsection 3.3.1. As well as the vector perturbations, there are two physical degrees of freedom. We then rewrite the system of the perturbation equations into a form including only the background parameter α. In subsection 3.3.2, we show that m^2 is real. We summarize the analytic solution for $\alpha = 1$ in subsection 3.3.3. Using this result, we numerically solve the perturbation equations for $\alpha < 1$ by relaxation method in subsection 3.3.4.

3.3.1 Basic equations

For scalar perturbations, we can take an analog of the longitudinal gauge:

$$
\begin{aligned}
ds_6^2 &= r^2(1 + \Psi Y)\eta_{\mu\nu}dx^\mu dx^\nu + [1 + (\Phi_1 + \Phi_2)Y]\frac{dr^2}{f} + 2h_{(L)\phi}\partial_\mu Y dx^\mu d\phi \\
&\quad + [1 - (\Phi_1 + 3\Phi_2)Y]\, f d\phi^2, \\
A_M dx^M &= a_r Y dr + (A + a_\phi Y)d\phi,
\end{aligned}
\tag{3.41}
$$

where perturbations are specified by the functions $\{\Psi,\ \Phi_1,\ \Phi_2,\ h_{(L)\phi},\ a_r,\ a_\phi\}$ of r and the harmonics $Y \equiv \exp(ik_\mu x^\mu)$. As for gauge fixing, see Appendix A.2. The $r\phi$-component of the Einstein equation implies that $h_{(L)\phi} = Cf(r)$, where C is an arbitrary constant. By using the residual gauge freedom $\tilde{C}$ in (A.37), we can set $C = 0$. Thus,

$$
h_{(L)\phi} = 0.
\tag{3.42}
$$

The r-component of the Maxwell equation and the (LL)-, $(L)r$- and $(L)\phi$-components of the Einstein equations give

$$
\begin{aligned}
a_r &= 0, \\
\Psi &= \Phi_2, \\
A'a_\phi &= \frac{1}{2r^2}(fr^2\Phi_1)' + f'\Phi_2,
\end{aligned}
\tag{3.43}
$$

where a prime denotes derivative with respect to r. The remaining equations are reduced to

$$
\begin{aligned}
\Phi_1'' + 2\left(\frac{f'}{f} + \frac{5}{r}\right)\Phi_1' - \frac{4\Lambda_6}{f}(\Phi_1 + \Phi_2) + \frac{m^2}{r^2 f}\Phi_1 &= 0, \\
\Phi_2'' + \frac{4}{r}\Phi_2' + \frac{m^2}{2r^2 f}(\Phi_1 + 2\Phi_2) &= 0,
\end{aligned}
\tag{3.44}
$$

where $m^2 \equiv -\eta^{\mu\nu}k_\mu k_\nu$.

With these equations, it is straightforward to show that linear perturbations of R, $R_{M'N'}^{MN}R_{MN}^{M'N'}$, $R^{;M}R_{;M}$ and $R^{KLMN;M'}R_{KLMN;M'}$ are independent linear combinations of $f\Phi_1$, $(f\Phi_1)'$, Φ_2 and $f'\Phi_2' - (m^2/2r^2)\Phi_1$ and that the matrix made of the coefficients remains regular and invertible in the $r \to r_\pm$ limit. Thus, $f\Phi_1$, $(f\Phi_1)'$, Φ_2 and $f'\Phi_2' - (m^2/2r^2)\Phi_1$ must be regular on the boundaries.

The correct boundary conditions can be obtained either by using the formalism developed in Sendouda et al. [28] or by setting the coefficients of $1/f$ in (3.44) to zero

on the boundary, where f vanishes. In the latter method, we obtain

$$
\left. 2f'\Phi'_1 - 4\Lambda_6(\Phi_1 + \Phi_2) + \frac{m^2}{r^2}\Phi_1 \right|_{r \to r_\pm} = 0,
$$

$$
\left. \frac{m^2}{2r^2}(\Phi_1 + 2\Phi_2) \right|_{r \to r_\pm} = 0. \tag{3.45}
$$

By using the boundary conditions, it is shown that Φ_1 and Φ_2 have regular Taylor expansion w.r.t. $r - r_\pm$. In practice, it is useful to note that for given values of $\Phi_1(r_\pm)$ and $\Phi'_2(r_\pm)$,

$$
\Phi_2(r_\pm) = -\frac{1}{2}\Phi_1(r_\pm),
$$

$$
\Phi'_1(r_\pm) = \left(\Lambda_6 - \frac{m^2}{2r_\pm^2} \right) \frac{\Phi_1(r_\pm)}{f'(r_\pm)},
$$

$$
\Phi''_2(r_\pm) = -\frac{4}{r_\pm}\Phi'_2(r_\pm) + \frac{m^2}{2r_\pm^2 f'(r_\pm)} \left[\left(\Lambda_6 - \frac{m^2}{2r_\pm^2} \right) \frac{\Phi_1(r_\pm)}{f'(r_\pm)} + 2\Phi'_2(r_\pm) \right]. \tag{3.46}
$$

The above derivation of the boundary condition and the Taylor expansion is simple and easy to follow. An alternative and more rigorous derivation is also possible by using the formalism developed in Sendouda et al. [28]. Following the formalism, the straightforward calculation gives

$$
\left. \left(\sqrt{f}f' \right)' \delta r - \frac{1}{2}f^{3/2}f'h_{rr} + \sqrt{f}h'_{\phi\phi} \right|_{r \to r_\pm} = 0,
$$

$$
\left. \left[\sqrt{f}\left(\sqrt{f}f' \right)' \right]' \delta r - f^{3/2}\left(\sqrt{f}f' \right)' h_{rr} + \sqrt{f}(\sqrt{f}h'_{\phi\phi})' - \frac{1}{2}ff'(fh_{rr})' \right|_{r \to r_\pm} = 0,
$$

$$
A'\delta r + a_\phi|_{r \to r_\pm} = 0 \tag{3.47}
$$

where $h_{rr} = (\Phi_1 + \Phi_2)/f$, $h_{\phi\phi} = -(\Phi_1 + 3\Phi_2)f$ and δr is defined by

$$
f'\delta r + h_{\phi\phi} = 0. \tag{3.48}
$$

By using the regularity of $f\Phi_1$, $(f\Phi_1)'$, Φ_2 and $f'\Phi'_2 - (m^2/2r^2)\Phi_1$ at $r = r_\pm$, the boundary conditions are reduced to

$$
f\Phi_1|_{r=r_\pm} = 0, \quad 2f'\Phi_2 + (f\Phi_1)'|_{r=r_\pm} = 0. \tag{3.49}
$$

This is actually equivalent to (3.45). Indeed, the same Taylor expansion, i.e. (3.46), follows also from (3.49).

22

Heretofore, the system of the above differential equations and the boundary conditions can be rewritten into a form which includes only the parameter α. In terms of $\bar{r}$, the Eq.(3.44) becomes

$$\partial_{\bar{r}}^2 \Phi_1 + 2\left(\frac{\partial_{\bar{r}}\bar{f}}{\bar{f}} + 5\frac{\beta_-}{\beta_-\bar{r}+\beta_+}\right)\partial_{\bar{r}}\Phi_1 - \frac{\Lambda_6}{\bar{f}}(\Phi_1+\Phi_2) + \frac{\tilde{m}^2}{(\beta_-\bar{r}+\beta_+)^2\bar{f}}\Phi_1 = 0,$$

$$\partial_{\bar{r}}^2 \Phi_2 + 4\frac{\beta_-}{\beta_-\bar{r}+\beta_+}\partial_{\bar{r}}\Phi_2 + \frac{\tilde{m}^2}{2(\beta_-\bar{r}+\beta_+)^2\bar{f}}(\Phi_1+2\Phi_2) = 0. \tag{3.50}$$

The function $\bar{f}$ and thus the Eq.(3.50) include only the parameter α. For numerical calculations in subsection 3.3.4, we use the first two equations in (3.46) as boundary conditions for Φ_1 and Φ_2. These boundary conditions are also rewritten in terms of $\bar{r}$, $\tilde{m}$ and α:

$$\Phi_1 + 2\Phi_2|_{\bar{r}=\pm1} = 0, \quad \partial_{\bar{r}}\Phi_1 = \left(\Lambda_6 - \frac{\tilde{m}^2}{2}\alpha^{\pm1}\right)\frac{1}{4\partial_{\bar{r}}\bar{f}}\Phi_1\bigg|_{\bar{r}=\pm1}. \tag{3.51}$$

Thus, to show the dynamical stability of scalar type perturbations, we calculate the spectrum of $\tilde{m}^2$ as a function of α by using (3.50) with (3.51), and show that $\tilde{m}^2$ is non-negative throughout.

As stated in the beginning of this subsection, we need to follow the four steps (i)-(iv) to show the stability. The first step (i) has not yet been considered at all, while the set of differential equations (3.50) with (3.51) is simple enough and expected to be useful for the remaining steps (ii)-(iv). Here we note that the Eq.(3.44) is not a manifestly self-adjoint system and, thus, we do not yet know whether the eigenvalue $\tilde{m}^2$ is real or not. In the next subsection 3.3.2 we shall reduce the system of differential equations to a manifestly self-adjoint one and show that $\tilde{m}^2$ is indeed real. After that, we shall come back to the equations (3.50) with (3.51) again and perform numerical calculations.

3.3.2 Reality of m^2 for scalar perturbation

To show the reality of m^2 for scalar perturbations, we suppose that $m^2 \neq 0$ and adopt the following gauge:

$$ds_6^2 = r^2(1+Q_1Y)\eta_{\mu\nu}dx^\mu dx^\nu + 2\mathcal{B}V_{(L)\mu}dx^\mu dr + (1+\mathcal{A}Y)\frac{dr^2}{f} + (1-3Q_1Y)fd\phi^2,$$

$$A_M dx^M = \mathcal{C}Y dr + (A+Q_2Y)d\phi, \tag{3.52}$$

where Q_1, Q_1, $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are functions of r. In this gauge the branes are at $r = r_\pm + 3ff'Q_1Y|_{r=r_\pm}$. Note that the corresponding metric in the five-dimensional

Einstein frame after reducing the ϕ direction is

$$ds^2_{5(E)} = r^2 f^{1/3}\eta_{\mu\nu}dx^\mu dx^\nu + 2f^{1/3}\mathcal{B}V_{(L)\mu}dx^\mu dr + [1 + (\mathcal{A} - Q_1)Y]\frac{dr^2}{f^{2/3}}. \qquad (3.53)$$

Thus, in this gauge a constant-r hypersurface in the five-dimensional Einstein frame is flat. Note also that the $(\phi\phi)$-component of the six-dimensional metric and the ϕ-component of the $U(1)$ field behave as scalar fields in five-dimension after reducing the ϕ direction. Therefore, Q_1 and Q_2 represent perturbations of the scalar fields on the flat hypersurface in the five-dimensional Einstein frame.

In the analysis of cosmological perturbations in the four-dimensional Einstein theory, the so called Mukhanov variables play important roles. The Mukhanov variables represent perturbations of the scalar fields on flat hypersurfaces, and thus, are analogous to our variables Q_1 and Q_2. Therefore, it is expected that the above gauge choice simplifies the analysis of perturbations.

The r-component of the Maxwell equation and the (LL)- and $(L)r$-components of the Einstein equations give the following algebraic equations.

$$\mathcal{C} = 0,$$
$$\frac{m^2}{r^2}\mathcal{B} = \frac{2rf'}{6f+rf'}Q'_1 - \frac{2rA'}{6f+rf'}Q'_2 - \frac{12(\Lambda_6 r^2 + 9f)(2f + rf') + r^2 f'^2}{(6f+rf')^2}\frac{Q_1}{r} + \frac{8\Lambda_6 r^3 A'}{(6f+rf')^2}\frac{Q_2}{r},$$
$$\mathcal{A} = \frac{3(2f - rf')}{6f+rf'}Q_1 + \frac{4rA'}{6f+rf'}Q_2, \qquad (3.54)$$

where a prime denotes derivative with respect to r and $m^2 = -\eta^{\mu\nu}k_\mu k_\nu$. The remaining equations are reduced to

$$\boldsymbol{L}\boldsymbol{Q} = m^2\boldsymbol{\Omega}\boldsymbol{Q}, \qquad (3.55)$$

where

$$\boldsymbol{Q} = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}, \qquad (3.56)$$

and

$$\boldsymbol{L} = \partial_r \boldsymbol{\alpha}\partial_r + \partial_r\boldsymbol{\beta} + \boldsymbol{\beta}\partial_r + \boldsymbol{\gamma},$$
$$\boldsymbol{\alpha} = r^4\begin{pmatrix} 3f & 0 \\ 0 & 1 \end{pmatrix},$$
$$\boldsymbol{\beta} = \frac{3}{2}r^4 A'\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\boldsymbol{\gamma} = -\frac{16r^6\Lambda_6 A'^2}{(6f+rf')^2}\begin{pmatrix} a & b \\ b & 1 \end{pmatrix},$$

24

$$\boldsymbol{\Omega} = \frac{r^2}{f}\begin{pmatrix} 3f & 0 \\ 0 & 1 \end{pmatrix},$$

$$a = -\frac{9f(2f+f')}{r^2 A'^2} + \frac{9(2\Lambda_6 + A'^2)}{32\Lambda_6 r^2 A'^2},$$

$$b = \frac{3(2f - rf')}{4rA'}. \tag{3.57}$$

With these equations, the linear perturbations of R, $R^{MN}_{M'N'} R^{M'N'}_{MN}$, $R^{;M} R_{;M}$ and $R^{KL}_{MN;M'} R^{MN;M'}_{KL}$ at the positions of the branes are independent linear combinations of Q_1, Q_2, Q'_1 and Q'_2 evaluated at $r = r_\pm$ [1]. The matrix made of the coefficients is regular and invertible. Thus, Q_1, Q_2, Q'_1 and Q'_2 must be regular at the positions of branes. With this regularity condition, it is shown by using the formalism developed in ref. [28] that the boundary condition at $r = r_\pm$ is

$$fQ_1|_{r=r_\pm} = Q_2|_{r=r_\pm} = 0. \tag{3.58}$$

Alternatively, the same boundary condition can be obtained from the boundary condition (3.45) and the relation (3.59) below. It is easy to show that the operator $\boldsymbol{L}$ with this boundary condition is hermite and that m^2 is real unless $Q_1 = Q_2 = 0$ everywhere in the interval $r_- \leq r \leq r_+$.

The Mukhanov-type variables (Q_1, Q_2) can be written in terms of the metric variables (Φ_1, Φ_2) in the analog of the longitudinal gauge. We have shown that m^2 is real unless the former variables vanish everywhere in the interval $r_- \leq r \leq r_+$. On the other hand, we have used the latter variables in the numerical calculations. The relations between the sets of variables are

$$Q_1 = \Phi_2 + \frac{2f\Phi_1}{6f + rf'},$$

$$Q_2 = a_\phi + \frac{frA'\Phi_1}{6f + rf'}, \tag{3.59}$$

where it is understood that a_ϕ is expressed in terms of Φ_1 and Φ_2. Equivalently, (Φ_1, Φ_2) are written in terms of (Q_1, Q_2) as

$$\Phi_1 = (6f + rf')\frac{\mathcal{B}}{r},$$

---

[1] Note that the positions of the branes are $r = r_\pm + 3f'fQ_1Y|_{r=r_\pm}$ and we do not know a priori whether fQ_1 vanishes at $r = r_\pm$ or not. Thus, for example, the linear perturbation of R at the positions of the branes is $\delta R + 3R^{(0)'} f'fQ_1Y$ evaluated at $r = r_\pm$, where $R^{(0)}$ and δR are the background value and the linear perturbation of the Ricci scalar, respectively. Similar statements hold also for the linear perturbations of $R^{MN}_{M'N'} R^{M'N'}_{MN}$, $R^{;M} R_{;M}$ and $R^{KLMN;M'} R_{KLMN;M'}$ at the positions of the branes.

$$\Phi_2 \;=\; Q_1 - \frac{2f\mathcal{B}}{r}, \tag{3.60}$$

where it is understood that $\mathcal{B}$ is expressed in terms of Q_1 and Q_2. Therefore, if $Q_1 = Q_2 = 0$ everywhere in the interval $r_- \le r \le r_+$ then $\Phi_1 = \Phi_2 = 0$ everywhere in the interval $r_- \le r \le r_+$. Therefore, we have shown that m^2 is real unless $\Phi_1 = \Phi_2 = 0$ everywhere in the interval $r_- \le r \le r_+$.

3.3.3 Analytic solution for $\alpha = 1$

The perturbation equations given in the subsection 3.3.1 can be analytically solved for $\alpha = 1$ as shown in Sendouda et al.[28]. Here we summarize the solution obtained there. By taking the $\alpha \to 1$ limit of the equations (3.50) and (3.51) we get

$$
\begin{aligned}
\left(1 - \bar{r}^2\right)\partial_{\bar{r}}^2 \Phi_1 - 4\bar{r}\partial_{\bar{r}}\Phi_1 - 2(\Phi_1 + \Phi_2) + \mu^2 \Phi_1 &= 0, \\
\left(1 - \bar{r}^2\right)\partial_{\bar{r}}^2 \Phi_2 + \frac{\mu^2}{2}\left(\Phi_1 + 2\Phi_2\right) &= 0,
\end{aligned} \tag{3.61}
$$

and

$$\Phi_1 + 2\Phi_2\big|_{\bar{r}=\pm 1} = 0, \quad \left(1 - \bar{r}^2\right)\partial_{\bar{r}}\Phi_1\big|_{\bar{r}=\pm 1} = 0, \tag{3.62}$$

where $\mu^2 \equiv \tilde{m}^2/2\Lambda_6$. The solution for the zero mode can be excluded using the first condition of (3.62) and the regularity of the variables at the boundaries [28]. In the following, we consider the case of $\mu^2 \neq 0$.

The differential equations are combined to

$$\partial_{\bar{r}}[(1 - \bar{r}^2)\partial_{\bar{r}}F_\pm] + \lambda_\mp F_\pm = 0, \tag{3.63}$$

where

$$
\begin{aligned}
F_\pm &= \partial_{\bar{r}}[(1 - \bar{r}^2)\partial_{\bar{r}}\Phi_2] + \lambda_\pm \Phi_2, \tag{3.64} \\
\lambda_\pm &= \mu^2 + 1 \pm \sqrt{3\mu^2 + 1}. \tag{3.65}
\end{aligned}
$$

If $\mu^2 \neq -1/3$, the solution is obtained as

$$
\begin{aligned}
\Phi_2 &= C_+ P_{\nu_+} + C_- P_{\nu_-} + D_+ Q_{\nu_+} + D_- Q_{\nu_-}, \tag{3.66} \\
\Phi_1 &= -\frac{2}{\mu^2}\{\partial_{\bar{r}}[(1 - \bar{r}^2)\partial_{\bar{r}}\Phi_2] + 2\bar{r}\partial_{\bar{r}}\Phi_2 + \Phi_2\}, \tag{3.67}
\end{aligned}
$$

where indices $\nu_\pm$ are determined by

$$\nu_\pm(\nu_\pm + 1) = \lambda_\pm = \mu^2 + 1 \pm \sqrt{3\mu^2 + 1}. \tag{3.68}$$

The first boundary condition is expanded around $\bar{r} = 1$ as

$$
\begin{aligned}
\Phi_1 + 2\Phi_2 \sim\ & 2\mu^{-2}(D_+ + D_-)(1-\bar{r})^{-1} \\
& -\mu^{-2}[(\lambda_+ - 1)D_+ + (\lambda_- - 1)D_-] \\
& +\mathcal{O}(1-\bar{r}).
\end{aligned} \tag{3.69}
$$

This means $D_+ = D_- = 0$, since $\lambda_+ \neq \lambda_-$ for $\mu^2 \neq -1/3$. Next we expand it around $\bar{r} = -1$, then

$$
\begin{aligned}
\Phi_1 + 2\Phi_2 \sim\ & 4\pi^{-1}\mu^{-2}(C_+ \sin\nu_+\pi + C_- \sin\nu_-\pi)(1+\bar{r})^{-1} \\
& +2\pi^{-1}\mu^{-2}[(\lambda_+ - 1)C_+ \sin\nu_+\pi + (\lambda_- - 1)C_- \sin\nu_-\pi] \\
& +\mathcal{O}(1+\bar{r}).
\end{aligned} \tag{3.70}
$$

This implies that non-trivial solutions can exist if $\nu_+ \in \mathbb{Z}$ or $\nu_- \in \mathbb{Z}$. With these choices of the parameters, we can confirm that the second boundary condition in (3.62) is satisfied. In general we can choose non-negative $\nu_\pm$'s so that these conditions are explicitly written down as

$$
\nu_\pm = \frac{-1 + \sqrt{4\mu^2 + 5 \pm 4\sqrt{3\mu^2 + 1}}}{2} = 0, 1, 2, \ldots. \tag{3.71}
$$

Therefore the KK mass spectrum for scalar perturbation is obtained. The case of $\mu^2 \neq -1/3$ can be excluded[28].

3.3.4 Numerical solution of KK modes

Here we obtain the first few KK modes of scalar type perturbations by numerically solving the perturbed Einstein equation (3.50) and the junction condition (3.51). We rewrite the system of two second order differential equations (3.50) to a system of four first order differential equations by defining $\partial_{\bar{r}}\Phi_1$ and $\partial_{\bar{r}}\Phi_2$ as well as Φ_1 and Φ_2 as dependent variables. Here, we take $M = 51$. We solve the problem while changing α from 1 to 0, each time with a slightly different value of α. For $\alpha = 1$, the analytic solutions presented in the previous subsection are used as trial solutions.

Figure 6 and 7 show the first four KK mode solutions of Φ_1 and Φ_2 for $\alpha = 1.0$ and 0.31. The normalization is determined by using the generalized Klein-Gordon norm (A.64) given in the Appendix A.4. In terms of the coordinate $(\bar{r}, \varphi)$, the equation (A.64) is written as

$$
(\tilde{\Phi}, \tilde{\Psi})_{KG} = M_6^4 (2\pi)^3 (\omega_1 + \omega_2)\delta^3(\mathbf{k}_1 - \mathbf{k}_2)e^{-i(\omega_1 - \omega_2)t}\frac{\Delta\varphi r_+ r_-}{2\Lambda_6}
$$

$$
\times \left\{ \int_{-1}^{1} \frac{d\bar{r}}{2\left(8\beta_-(\beta_-\bar{r}+\beta_+)\partial_{\bar{r}}\bar{f}+24\beta_-^2\bar{f}+\Lambda_6(\beta_-\bar{r}+\beta_+)^2\right)} \right.
$$

$$
\times \left[\frac{1}{8}(\tilde{m}_1^2+\tilde{m}_2^2)(\beta_-\bar{r}+\beta_+)^2\Phi_1\Psi_1^* + \frac{(\beta_-\bar{r}+\beta_+)^4}{2}\partial_{\bar{r}}\bar{f}(\Psi_2^*\partial_{\bar{r}}\Phi_1+\Phi_2\partial_{\bar{r}}\Psi_1^*) \right.
$$

$$
+ \frac{(\beta_-\bar{r}+\beta_+)^2}{4}\left(13\beta_-^2\bar{f}+3\Lambda_6\frac{(\beta_-\bar{r}+\beta_+)^2}{4}\right)\Phi_1\Psi_1^*
$$

$$
+(\beta_-\bar{r}+\beta_+)^2\left(2\beta_-(\beta_-\bar{r}+\beta_+)\partial_{\bar{r}}\bar{f}+12\beta_-^2\bar{f}+3\Lambda_6\frac{(\beta_-\bar{r}+\beta_+)^2}{4}\right)\Phi_2\Psi_2^*
$$

$$
+\frac{(\beta_-\bar{r}+\beta_+)^2}{4\bar{f}}\left((\beta_-\bar{r}+\beta_+)^2\partial_{\bar{r}}\bar{f}^2+10\beta_-(\beta_-\bar{r}+\beta_+)\bar{f}\partial_{\bar{r}}\bar{f}+24\beta_-^2\bar{f}^2\right.
$$

$$
\left.+3\Lambda_6\bar{f}\frac{(\beta_-\bar{r}+\beta_+)^2}{4}\right)(\Phi_1+2\Phi_2)(\Psi_1^*+2\Psi_2^*)\Bigg]
$$

$$
+\left. \left[\frac{3(\beta_-\bar{r}+\beta_+)^3\partial_{\bar{r}}\bar{f}\Phi_1\Psi_1^*}{16(8\beta_-\partial_{\bar{r}}\bar{f}+\Lambda_6(\beta_-\bar{r}+\beta_+))}\right]_{\bar{r}=-1}^{\bar{r}=1} \right\}
$$

$$
\equiv \; M_6^4(2\pi)^3(\omega_1+\omega_2)\delta^3(\mathbf{k}_1-\mathbf{k}_2)e^{-i(\omega_1-\omega_2)t}\frac{\Delta\varphi r_+r_-}{2\Lambda_6}\left(\Phi_1,\Phi_2\Big|\Psi_1,\Phi_2\right). \quad (3.72)
$$

We normalize the solution by

$$
\left(\Phi_1^{(n_1)},\Phi_2^{(n_1)}\Big|\Phi_1^{(n_2)},\Phi_2^{(n_2)}\right)=\delta_{n_1 n_2}, \quad (3.73)
$$

where superscript n_k means that $\Phi_1^{(n_k)}$ and $\Phi_2^{(n_k)}$ are the solutions of scalar perturbations with eigenvalue $m_{n_k}^2$.

Finally, we show the spectrum of m_+^2 of the first four KK modes as a function of α in figure 8. We find that m_+^2 is non-negative for the entire range of α. Therefore, the background spacetime is dynamically stable in the scalar-type sector. The behavior of m_+^2 in $\alpha \to 0$ limit is also similar to the cases of vector and tensor perturbations. It remains finite in this limit, which is consistent with our previous work [12], where we obtained the energy scale at which the correction to the effective Friedmann equation on the brane appears.

4 Summary and Discussion

In this paper, we have considered the dynamical stability of the six-dimensional brane world model with warped flux compactification recently found by the authors. This solution captures essential features of the warped flux compactification, including warped geometry, compactification, a magnetic flux, and one or two 3-brane(s). For
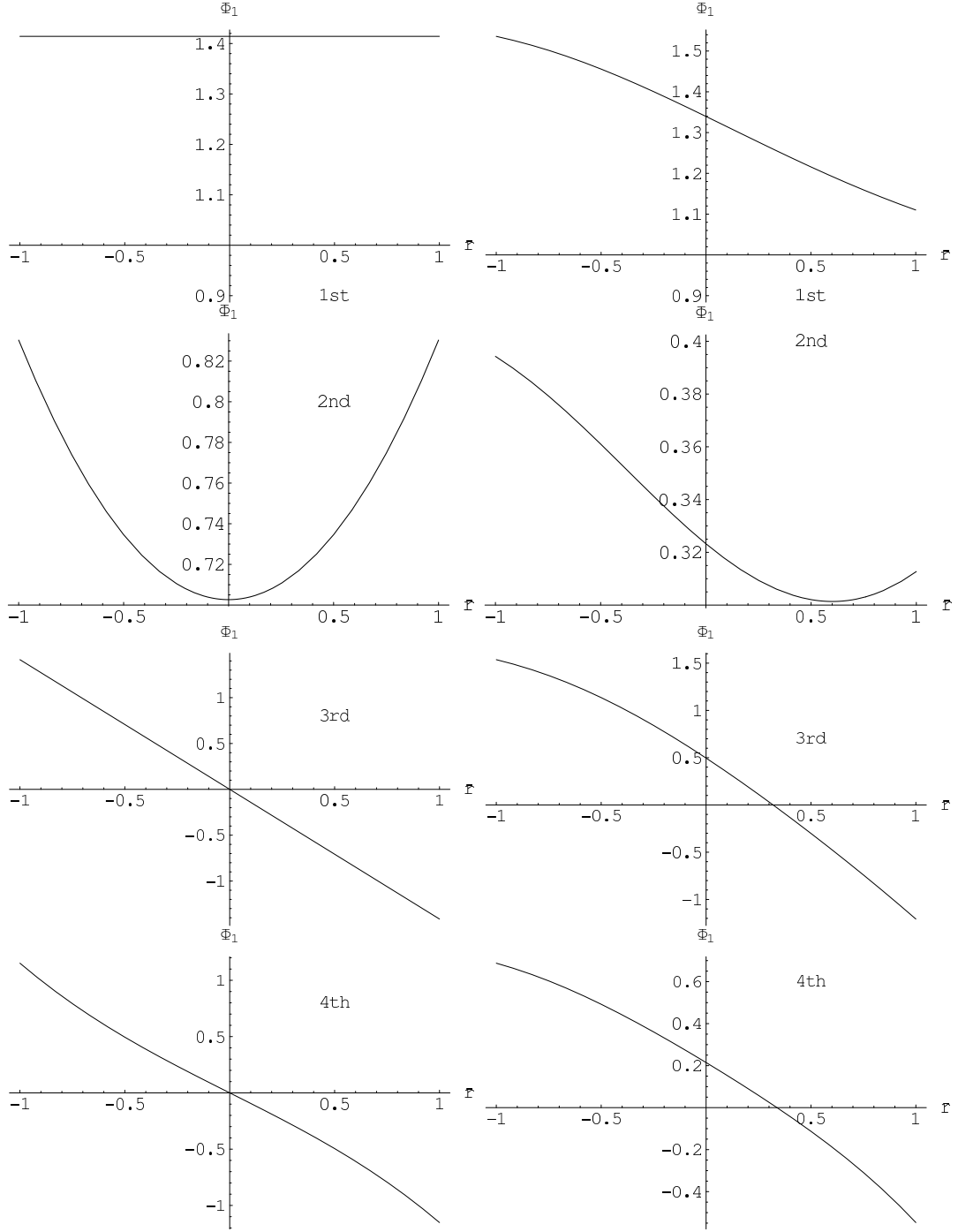
Figure 6: The solution Φ_1 of the first four KK modes for $\alpha = 1$ (left) and $\alpha = 0.31$ (right). The normalization is determined by using the generalized Klein-Gordon norm (see the text). Number of points of the mesh is taken to be 51.
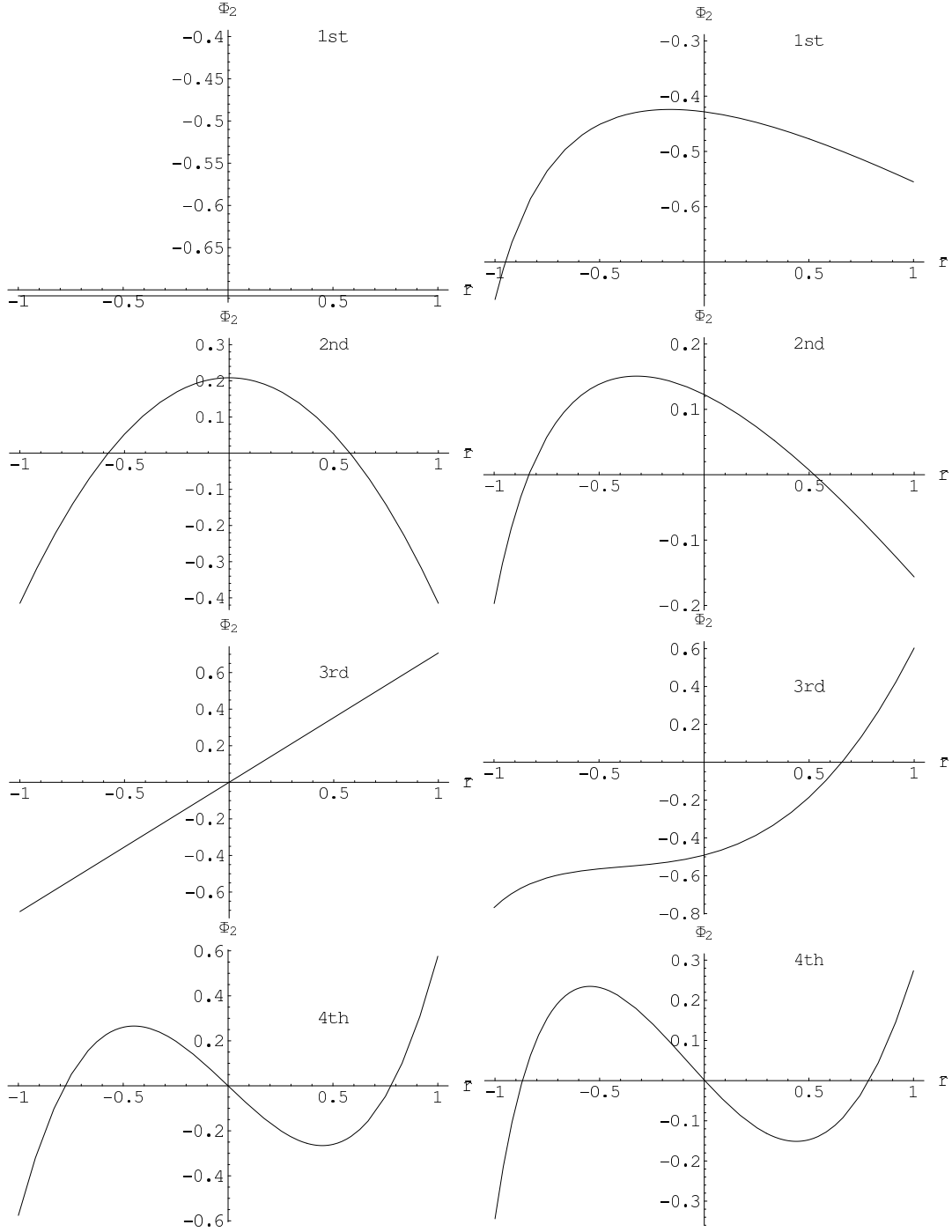
Figure 7: The solution Φ_2 of the first four KK modes for $\alpha = 1$ (left) and $\alpha = 0.31$ (right). The normalization is determined by using the generalized Klein-Gordon norm (see the text). Number of points of the mesh is taken to be 51.
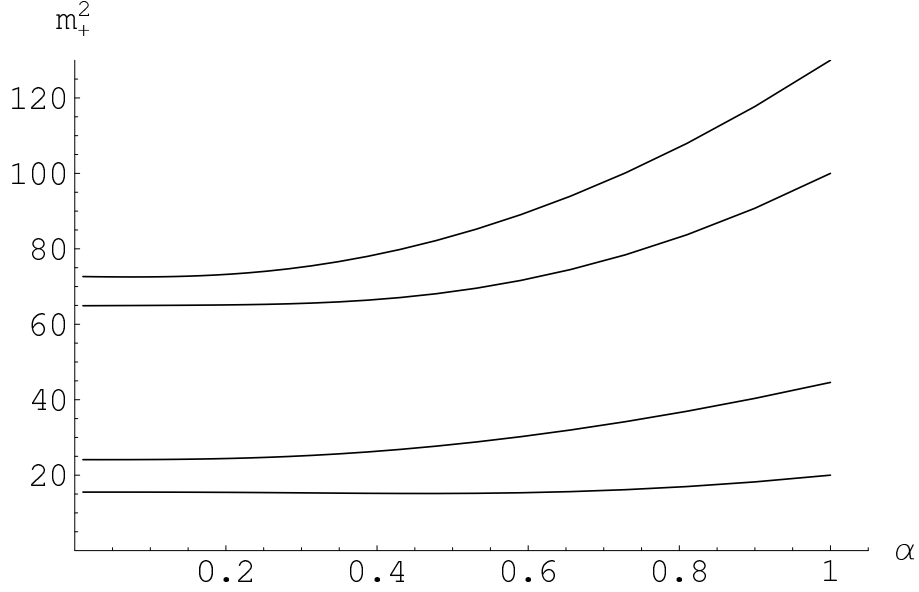
30

Figure 8: The spectrum of m_+^2 for scalar perturbations as a function of α.

simplicity we have set the four-dimensional cosmological constant to zero, and re-
stricted to linear perturbations with the axisymmetry corresponding to the rotation
in the two-dimensional bulk. We have expanded perturbations by scalar-, vector- and
tensor-type harmonics of the four-dimensional Minkowski spacetime and analyzed
each type separately.

The perturbations were labeled by its type and values of mass squared $m^2 = -\eta^{\mu\nu}k_\mu k_\nu$. To study the stability, we have utilized the fact that for any perturbation
type, the perturbation equations are reduced to an eigenvalue problem with eigenvalue
m^2. We regarded the background spacetime to be dynamically stable if the spectrum
of m^2 is non-negative. This question was treated in each sector through the following
steps.

  (i) We have shown the reality of m^2.

 (ii) We have rewritten the systems of the perturbation equations into a form which
      depends on background parameters through just one parameter α. $(0 \le \alpha \le 1)$

(iii) In the $\alpha \to 1$ limit we have analytically solved the perturbation equations and
      have shown that the spectrum of m^2 is non-negative.

(iv) We have numerically evaluated how each eigenvalue m^2 changes as α moves from
     1 to 0, and have shown that the spectrum remains non-negative throughout.

31

For $\alpha < 1$, we had to numerically solve the systems of the perturbation equations many times, each time with a slightly different values of α. In such a situation, relaxation method is useful, and thus we have employed this method. When α is slightly changed, the previous solution will be a good initial guess, and relaxation works well. Since the perturbations can be analytically solved in all the sectors for $\alpha = 1$, we have started the sequence of numerical computations from $\alpha = 1 - \Delta\alpha$ by using the analytic solutions as the initial guess, where $\Delta\alpha$ is a small positive number.

The mass squared that has physical meaning is $m_\pm^2 \equiv -r_\pm^{-2}\eta^{\mu\nu}k_\mu k_\nu$, which is the one observed on the brane at $r = r_\pm$. We have shown the spectra of $m_\pm^2$ as a function of α in each sector, and found that m_+^2 is non-negative for the entire range of α, $[0, 1]$. Therefore, the background spacetime we consider is dynamical stable in each sector. We have found that there are zero modes only in the tensor sector, corresponding to the four-dimensional gravitons.

Another remarkable feature of the spectra of $m_\pm^2$ is that they remain finite in $\alpha \to 0$ limit, that is, $m_+^2 \propto \alpha^0$ and $m_-^2 \propto \alpha^{-2}$ for $\alpha \to 0$ in all the sectors. This result is consistent with our previous study [12], where we analyzed how the Hubble expansion rate $H_\pm$ on each brane changes when the brane tension changes. We also considered higher-order corrections of the effective Friedmann equation with respect to $H_\pm$. The result is that higher-order corrections appears when $H_\pm$ is larger than a critical value $H_{*\pm}$. For $\alpha \to 0$, we found that $H_{*\pm}$ behave as the equation (3.17). Since the higher-order corrections are caused by the KK modes, this energy scale corresponds to their mass squared. Thus, the behavior of $m_\pm^2$ we obtained is consistent with our previous result.

Having established the stability of the exact braneworld solution, there are many subjects for future research, including the recovery of 4-dimensional linearized Einstein gravity and corrections to it, the recovery of the 4-dimensional Friedmann equation, properties of black hole geometries [29], and so on.

Note added

After this paper was submitted for publication, we were notified that the exact solution considered by the authors had been already found in [30, 19] in advance of our previous paper [12].

Appendices

A.1 Harmonics in Minkowski spacetime

In this appendix we give definitions of scalar, vector and tensor harmonics in an n-dimensional Minkowski spacetime. Throughout this appendix, n-dimensional coordinates are x^μ ($\mu = 0, 1, \cdots, n-1$), $\eta_{\mu\nu}$ is the Minkowski metric, and all indices are raised and lowered by the Minkowski metric and its inverse $\eta^{\mu\nu}$.

A.1.1 Scalar harmonics

The scalar harmonics are given by

$$Y = \exp(ik_\rho x^\rho), \tag{A.1}$$

by which any function f can be expanded as

$$f = \int dk \; cY, \tag{A.2}$$

where c is a constant depending on k. Hereafter, k and dk are abbreviations of $\{k^\mu\}$ ($\mu = 0, 1, \cdots, n-1$) and $\prod_{\mu=0}^{n-1} dk^\mu$, respectively. We omit k in most cases.

A.1.2 Vector harmonics

In general, any vector field v_μ can be decomposed as

$$v_\mu = v_{(T)\mu} + \partial_\mu f, \tag{A.3}$$

33

where f is a function and $v_{(T)\mu}$ is a transverse vector field:

$$\partial^\mu v_{(T)\mu} = 0. \tag{A.4}$$

Thus, the vector field v_μ can be expanded by using the scalar harmonics Y and transverse vector harmonics $V_{(T)\mu}$ as

$$v_\mu = \int dk \left[c_{(T)} V_{(T)\mu} + c_{(L)} \partial_\mu Y \right]. \tag{A.5}$$

Here, $c_{(T)}$ and $c_{(L)}$ are constants depending on k, and the transverse vector harmonics $V_{(T)\mu}$ are given by

$$V_{(T)\mu} = u_\mu \exp(ik_\rho x^\rho), \tag{A.6}$$

where the constant vector u_μ satisfies the following condition.

$$k^\mu u_\mu = 0 \tag{A.7}$$

for $k^\mu k_\mu \neq 0$, and

$$\begin{aligned}
k^\mu u_\mu &= 0, \\
\tau^\mu u_\mu &= 0
\end{aligned} \tag{A.8}$$

for non-vanishing k_μ satisfying $k^\mu k_\mu = 0$, where τ^μ is an arbitrary constant time-like vector. For $k_\mu = 0$, the constant vector u^μ does not need to satisfy any of the above conditions. For the special case $k^\mu k_\mu = 0$, the second condition in (A.8) can be imposed by redefinition of $c_{(L)}$. Actually this condition is necessary to eliminate redundancy. Note that the number of independent vectors satisfying the above condition is $n-1$ for $k^\mu k_\mu \neq 0$ and $n-2$ for $k^\mu k_\mu = 0$ and that these numbers are equal to the numbers of physical degrees of freedom for massive and massless spin-1 fields in n-dimensions, respectively.

Because of the expansion (A.5), it is convenient to define longitudinal vector harmonics $V_{(L)\mu}$ by

$$V_{(L)\mu} \equiv \partial_\mu Y = ik_\mu Y. \tag{A.9}$$

A.1.3   Tensor harmonics

In general, a symmetric second-rank tensor field $t_{\mu\nu}$ can be decomposed as

$$t_{\mu\nu} = t_{(T)\mu\nu} + \partial_\mu v_\nu + \partial_\nu v_\mu + f\eta_{\mu\nu}, \tag{A.10}$$

where f is a function, v_μ is a vector field and $t_{(T)\mu\nu}$ is a transverse traceless symmetric tensor field:

$$
\begin{aligned}
t^\mu_{(T)\mu} &= 0, \\
\partial^\mu t_{(T)\mu\nu} &= 0.
\end{aligned}
\tag{A.11}
$$

Thus, the tensor field $t_{\mu\nu}$ can be expanded by using the scalar harmonics Y, the vector harmonics $V_{(T)}$ and $V_{(L)}$, and transverse traceless tensor harmonics $T_{(T)}$ as

$$
\begin{aligned}
t_{\mu\nu} &= \int dk \left[c_{(T)} T_{(T)\mu\nu} + c_{(LT)} (\partial_\mu V_{(T)\nu} + \partial_\nu V_{(T)\mu}) \right. \\
&\left. \quad + c_{(LL)} (\partial_\mu V_{(L)\nu} + \partial_\nu V_{(L)\mu}) + \tilde{c}_{(Y)} Y \eta_{\mu\nu} \right].
\end{aligned}
\tag{A.12}
$$

Here, $c_{(T)}$, $c_{(LT)}$, $c_{(LL)}$, and $\tilde{c}_{(Y)}$ are constants depending on k, and the transverse traceless tensor harmonics $T_{(T)}$ are given by

$$
T_{(T)\mu\nu} = s_{\mu\nu} \exp(ik_\rho x^\rho),
\tag{A.13}
$$

where the constant symmetric second-rank tensor $s_{\mu\nu}$ satisfies the following condition.

$$
\begin{aligned}
k^\mu s_{\mu\nu} &= 0, \\
s^\mu_\mu &= 0
\end{aligned}
\tag{A.14}
$$

for $k^\mu k_\mu \neq 0$, and

$$
\begin{aligned}
k^\mu s_{\mu\nu} &= 0, \\
s^\mu_\mu &= 0, \\
\tau^\mu s_{\mu\nu} &= 0
\end{aligned}
\tag{A.15}
$$

for non-vanishing k_μ satisfying $k^\mu k_\mu = 0$, where τ^μ is an arbitrary constant timelike vector. For $k_\mu = 0$, the constant tensor $s_{\mu\nu}$ does not need to satisfy any of the above conditions. For the special case $k^\mu k_\mu = 0$, the last condition in (A.15) can be imposed by redefinition of $c_{(LT)}$, $c_{(LL)}$ and $\tilde{c}_{(Y)}$. Actually this condition is necessary to eliminate redundancy. Note that the number of independent symmetric second-rank tensors satisfying the above conditions is $(n+1)(n-2)/2$ for $k^\mu k_\mu \neq 0$ and $n(n-3)/2$ for $k^\mu k_\mu = 0$ and that these numbers are equal to numbers of physical degrees of freedom for massive and massless spin-2 fields in n-dimensions, respectively.

Because of the expansion (A.12), it is convenient to define tensor harmonics $T_{(LT)}$, $T_{(LL)}$, and $T_{(Y)}$ by

$$
T_{(LT)\mu\nu} \equiv \partial_\mu V_{(T)\nu} + \partial_\nu V_{(T)\mu},
$$

$$= i(u_\mu k_\nu + u_\nu k_\mu)Y,$$

$$T_{(LL)\mu\nu} \equiv \partial_\mu V_{(L)\nu} + \partial_\nu V_{(L)\mu} - \frac{2}{n}\eta_{\mu\nu}\partial^\rho V_{(L)\rho}$$

$$= \left(-2k_\mu k_\nu + \frac{2}{n}k^\rho k_\rho \eta_{\mu\nu}\right) Y,$$

$$T_{(Y)\mu\nu} \equiv \eta_{\mu\nu}Y. \tag{A.16}$$

A.2 Gauge transformation of the perturbations

In this appendix we give gauge transformations of the perturbations of the metric and the $U(1)$ gauge field. The coordinate gauge transformation is of the form

$$x^M \to x^M + \bar{\xi}^M. \tag{A.17}$$

There is another kind of gauge transformation. The perturbations of the field strength δF_{MN} are not changed under gauge transformation of the gauge field,

$$\delta A_M \to \delta A_M + \partial_M \bar{\zeta}. \tag{A.18}$$

The gauge parameters $\bar{\xi}^M$ and $\bar{\zeta}$ can be expanded by the scalar and the vector harmonics as

$$\bar{\xi}_M dx^M = \left(\xi_{(T)}V_{(T)\mu} + \xi_{(L)}V_{(L)\mu}\right) dx^\mu + \xi_r Y dr + \xi_\phi Y d\phi, \tag{A.19}$$

$$\bar{\zeta} = \zeta Y, \tag{A.20}$$

where $\xi_{(T,L)}$, ξ_r, ξ_ϕ and ζ are supposed to depend only on r. Under the above gauge transformation, the perturbation variables transform as

$$\bar{h}_{(T)} = h_{(T)}, \tag{A.21}$$

$$\bar{h}_{(LT)} = h_{(LT)} - \xi_{(T)}, \tag{A.22}$$

$$\bar{h}_{(LL)} = h_{(LL)} - \xi_{(L)}, \tag{A.23}$$

$$\bar{h}_{(Y)} = h_{(Y)} - 2rf\xi_r + \frac{1}{2}k^\mu k_\mu \xi_{(L)}, \tag{A.24}$$

$$\bar{h}_{(T)r} = h_{(T)r} + \frac{2}{r}\xi_{(T)} - \xi'_{(T)}, \tag{A.25}$$

$$\bar{h}_{(L)r} = h_{(L)r} + \frac{2}{r}\xi_{(L)} - \xi'_{(L)} - \xi_r, \tag{A.26}$$

$$\bar{h}_{(L)\phi} = h_{(L)\phi} - \xi_\phi, \tag{A.27}$$

$$\bar{h}_{rr} = h_{rr} - 2\xi'_r - \frac{f'}{f}\xi_r, \tag{A.28}$$

36

$$\bar{h}_{r\phi} = h_{r\phi} - \xi'_\phi + \frac{f'}{f}\xi_\phi, \tag{A.29}$$

$$\bar{h}_{\phi\phi} = h_{\phi\phi} - ff'\xi_r, \tag{A.30}$$

$$\bar{a}_{(T)} = a_{(T)}, \tag{A.31}$$

$$\bar{a}_{(L)} = a_{(L)} + \zeta - \frac{A}{f}\xi_\phi, \tag{A.32}$$

$$\bar{a}_r = a_r + \partial_r\zeta - \frac{A}{f}\left(\xi'_\phi - \frac{f'}{f}\xi_\phi\right), \tag{A.33}$$

$$\bar{a}_\phi = a_\phi - fA'\xi_r. \tag{A.34}$$

Finally we summarize the gauge conditions employed in this paper. Of course, tensor type perturbations are gauge invariant from the beginning. For vector perturbations, we set $\bar{h}_{(LT)} = 0$ by taking $\xi_{(T)} = h_{(LT)}$. The master equations for scalar perturbations in the main part of section 3.3 are derived in the analog of the Longitudinal gauge where $\bar{h}_{(LL)} = \bar{h}_{(L)r} = \bar{h}_{r\phi} = 0$. This can be obtained by taking a gauge

$$\xi_{(L)} = h_{(LL)}, \tag{A.35}$$

$$\xi_r = h_{(L)r} + \frac{2}{r}h_{(LL)} - h'_{(LL)}, \tag{A.36}$$

$$\xi_\phi = \tilde{C}f(r) + f(r)\int^r dr'\frac{h_{r\phi}(r')}{f(r')}, \tag{A.37}$$

where $\tilde{C}$ is an arbitrary constant representing the residual gauge freedom. We use this residual gauge freedom to set $C = 0$ in subsection 3.3.1. In subsection 3.3.2, we show the reality of m^2 for scalar perturbations by adopting another gauge. This gauge corresponds to $\bar{h}_{(LL)} = 0$, $3f\bar{h}_{(Y)} + r^2\bar{h}_{\phi\phi} = 0$ and $\bar{h}_{r\phi} = 0$, which can be set by

$$\xi_{(L)} = h_{(LL)}, \tag{A.38}$$

$$\xi_r = \frac{3fh_{(Y)} + r^2h_{\phi\phi} + 3k^\mu k_\mu h_{(LL)}}{rf(6f + rf')}, \tag{A.39}$$

$$\xi_\phi = \tilde{C}f(r) + f(r)\int^r dr'\frac{h_{r\phi}(r')}{f(r')}, \tag{A.40}$$

where $\tilde{C}$ is again an arbitrary constant representing the residual gauge freedom. For the U(1) gauge field, we set $\bar{a}_{(L)} = 0$ by $\zeta = a_{(L)}$ both in the main part of section 3.3 and in subsection 3.3.2.

A.3 Relaxation method

In the following we explain relaxation method in detail [31]. First of all, we rewrite a system of second-order differential equations to a system of first-order differential equations of the form

$$\frac{dy_i\left(\bar{r}\right)}{d\bar{r}} = g_i\left(\bar{r}, y_1, y_2, \cdots, y_N, \tilde{m}^2\right) \quad (i = 1, 2, \cdots, N), \tag{A.41}$$

where y_i denotes one of N dependent functions. For example, $y_1 = h$ and $y_2 = \partial_{\bar{r}} h$ for tensor perturbations. When we numerically solve the equations, one of the dependent functions has to be fixed at some $\bar{r}$ in an arbitrary manner. (When we plot the solution, we use another normalization.) Thus, these dependent functions have to satisfy $N + 1$ boundary conditions instead of just N. The problem is overdetermined and in general there is no solution for arbitrary values of $\tilde{m}^2$. For certain special values of $\tilde{m}^2$, the Eq.(A.41) does have a solution. Such $\tilde{m}^2$ is the eigenvalue. It is convenient to reduce this problem to the standard case by introducing a new dependent variable

$$y_{N+1} \equiv \tilde{m}^2 \tag{A.42}$$

and another differential equation

$$\frac{dy_{N+1}}{d\bar{r}} = 0. \tag{A.43}$$

Next, we replace the differential equations with finite-difference equations. We first define a mesh by a set of $k = 1, 2, \cdots, M$ points at which we supply values for the independent variable $\bar{r}_k$. In particular, $\bar{r}_1 (= -1)$ is the initial boundary, and $\bar{r}_M (= 1)$ is the final boundary. We use the notation $\mathbf{y}_k$ to refer to the entire set of dependent variables $y_1, y_2, \cdots, y_{N+1}$ at point $\bar{r}_k$. At an arbitrary point k in the middle of the mesh, we approximate the set of five differential equations by algebraic relations of the form

$$0 = \mathbf{E}_k \equiv \mathbf{y}_k - \mathbf{y}_{k-1} - \left(\bar{r}_k - \bar{r}_{k-1}\right) \mathbf{g}_k\left(\bar{r}_k, \bar{r}_{k-1}, \mathbf{y}_k, \mathbf{y}_{k-1}\right), \quad k = 2, 3, \cdots, M. \tag{A.44}$$

The finite-difference equations labeled by $\mathbf{E}_k$ provide a total of $(N + 1)(M - 1)$ equations for the $(N + 1) M$ unknowns. The remaining $N + 1$ equations come from the boundary conditions. At the first boundary we have

$$0 = \mathbf{E}_1 \equiv \mathbf{B}\left(\bar{r}_1, \mathbf{y}_1\right) \tag{A.45}$$

while at the second boundary

$$0 = \mathbf{E}_{M+1} \equiv \mathbf{C}\left(\bar{r}_M, \mathbf{y}_M\right). \tag{A.46}$$

The vectors $\mathbf{B}$ and $\mathbf{C}$ have only n_2 and $n_1 = N + 1 - n_2$ nonzero components respectively, corresponding to number of the boundary conditions.

The solution of the Eq.(A.44), (A.45), and (A.46) consists of a set of variables $\mathbf{y}_k$ at the M points $\bar{r}_k$. The algorithm we now describe requires an initial guess for $\mathbf{y}_k$. We then determine increments $\Delta \mathbf{y}_k$ such that $\mathbf{y}_k + \Delta \mathbf{y}_k$ is an improved approximation to the solution. Equations for the increments are developed by expanding the Eq.(A.44) in first-order Taylor series with respect to $\Delta \mathbf{y}_k$. At an interior point, $k = 2, 3, \cdots, M$ this gives

$$
\begin{aligned}
0 = \mathbf{E}_k \left(\mathbf{y}_k + \Delta \mathbf{y}_k, \mathbf{y}_{k-1} + \Delta \mathbf{y}_{k-1} \right) &\sim \mathbf{E}_k \left(\mathbf{y}_k, \mathbf{y}_{k-1} \right) \\
&+ \sum_{n=1}^{N+1} \frac{\partial \mathbf{E}_k}{\partial y_{n,k-1}} \Delta y_{n,k-1} + \sum_{n=1}^{N+1} \frac{\partial \mathbf{E}_k}{\partial y_{n,k}} \Delta y_{n,k},
\end{aligned}
\tag{A.47}
$$

where $y_{n,k}$ is the value of y_n at the point $\bar{r}_k$. This provides $(N+1)(M-1)$ equations for $\Delta \mathbf{y}_k$. Similarly, the algebraic relations at the boundaries can be expanded in a first-order Taylor series for increments that improve the solution. Since $\mathbf{E}_1$ depends only on $\mathbf{y}_1$, we find at the first boundary:

$$
0 = \mathbf{E}_1 + \sum_{n=1}^{N+1} \frac{\partial \mathbf{E}_1}{\partial y_{n,1}} \Delta y_{n,1}.
\tag{A.48}
$$

At the second boundary,

$$
0 = \mathbf{E}_{M+1} + \sum_{n=1}^{N+1} \frac{\partial \mathbf{E}_{M+1}}{\partial y_{n,M}} \Delta y_{n,M}.
\tag{A.49}
$$

We again note that the Eq.(A.48) and (A.49) have only n_2 and $n_1 = N + 1 - n_2$ nonzero components. We thus have a set of $(N+1)M$ linear equations to be solved for the corrections $\Delta \mathbf{y}_k$, iterating until the corrections are sufficiently small.

A.4 The generalized Klein-Gordon norm

To normalize mode functions we use the generalized Klein-Gordon norm, whose definition can be easily read off from the kinetic term in the effective action for the corresponding physical degree of freedom. See, for example, Appendix A of ref. [32] for the definition and the motivation of the generalized Klein-Gordon norm.

A.4.1 Tensor perturbation

For tensor perturbation,

$$
ds_6^2 = r^2 \left[\eta_{\mu\nu} + h_{\mu\nu} \right] dx^\mu dx^\nu + \frac{dr^2}{f} + f d\phi^2,
$$

$$A_M dx^M \;=\; A d\phi, \tag{A.50}$$

where $h_{\mu\nu}$ is a symmetric, transverse and traceless four-dimensional tensor depending on $(x^\mu,\, r)$, the bulk action is expanded up to the second order in perturbation as

$$
\begin{aligned}
I_6 \;&=\; \frac{M_6^4}{2} \int d^6 x \sqrt{-g} \left(R - 2\Lambda_6 - \frac{1}{2} F^{MN} F_{MN} \right) \\
&=\; -\frac{M_6^4 \Delta\phi}{16} \int d^4 x \int dr\, r^2 \eta^{\rho\rho'} \eta^{\sigma\sigma'} \left[\eta^{\mu\nu} \partial_\mu h_{\rho\sigma} \partial_\nu h_{\rho'\sigma'} + f r^2 \partial_r h_{\rho\sigma} \partial_r h_{\rho'\sigma'} \right] \tag{A.51}
\end{aligned}
$$

Here, we have not written down the boundary term since it does not change the definition of the generalized Klein-Gordon norm. It is easy to check that the correct equation of motion is derived from this action.

From this form of the action we can read off the generalized Klein-Gordon norm as

$$(\Phi, \Psi)_{KG} \equiv -i \frac{M_6^4 \Delta\phi}{8} \int d^3 \mathbf{x} \int dr\, r^2 \eta^{\mu\mu'} \eta^{\nu\nu'} \left(\Phi_{\mu\nu} \partial_t \Psi^*_{\mu'\nu'} - \Psi^*_{\mu\nu} \partial_t \Phi_{\mu'\nu'} \right). \tag{A.52}$$

A.4.2 Vector perturbation

For vector perturbation, after fixing the gauge freedom $(h_{(LT)} = 0)$ and using the corresponding constraint equation (the (LT) component of the Einstein equation), the metric and the $U(1)$ field in the linearized level are written as

$$
\begin{aligned}
ds_6^2 \;&=\; r^2 \eta_{\mu\nu} dx^\mu dx^\nu + 2 \left[\frac{h_{r\mu}}{r^2 f} dr + h_{\phi\mu} d\phi \right] dx^\mu + \frac{dr^2}{f} + f d\phi^2, \\
A_M dx^M \;&=\; a_\mu dx^\mu + A d\phi, \tag{A.53}
\end{aligned}
$$

where a_μ and $h_{\phi\mu}$ are transverse four-dimensional vectors depending on $(x^\mu,\, r)$ and $h_{r\mu}$ is a transverse four-dimensional vector depending only on x^μ. The bulk action is expanded up to the second order in perturbation as

$$
\begin{aligned}
I_6 \;&=\; \frac{M_6^4}{2} \int d^6 x \sqrt{-g} \left(R - 2\Lambda_6 - \frac{1}{2} F^{MN} F_{MN} \right) \\
&=\; \frac{M_6^4 \Delta\phi}{4} \int d^4 x \int dr\, L, \tag{A.54}
\end{aligned}
$$

where

$$L = -\eta^{\rho\sigma} \left[2\eta^{\mu\nu} \partial_\mu a_\rho \partial_\nu a_\sigma + \frac{1}{f} \eta^{\mu\nu} \partial_\mu h_\rho \partial_\nu h_\sigma + 2 f r^2 \partial_r a_\rho \partial_r a_\sigma - 4 r^2 A' h_\rho \partial_r a_\sigma + r^2 \partial_r h_\rho \partial_r h_\sigma + 6 h_\rho h_\sigma \right]. \tag{A.55}$$

We have not written down the boundary term since it does not change the definition of the generalized Klein-Gordon norm. It is easy to check that the correct equations of motion are derived from this action.

From this form of the action we can read off the generalized Klein-Gordon norm as

$$(\tilde{\Phi}, \tilde{\Psi})_{KG} \equiv -i\frac{M_6^4 \Delta\phi}{2} \int d^3\mathbf{x} \int dr \eta^{\mu\nu} \left[\left(\tilde{\Phi}_{1\mu}\partial_t\tilde{\Psi}_{1\nu}^* - \tilde{\Psi}_{1\mu}^*\partial_t\tilde{\Phi}_{1\nu} \right) + \frac{r^4}{f} \left(\tilde{\Phi}_{2\mu}\partial_t\tilde{\Psi}_{2\nu}^* - \tilde{\Psi}_{2\mu}^*\partial_t\tilde{\Phi}_{2\nu} \right) \right],$$
(A.56)

where the solutions $\tilde{\Phi}$ and $\tilde{\Psi}$ are specified by $(\tilde{\Phi}_{1\mu}, \tilde{\Phi}_{2\mu})$ and $(\tilde{\Psi}_{1\mu}, \tilde{\Psi}_{2\mu})$, respectively, as

$$\tilde{\Phi}: \quad a_\mu = \frac{\tilde{\Phi}_{1\mu}}{\sqrt{2}}, \quad h_\mu = r^2\tilde{\Phi}_{2\mu},$$

$$\tilde{\Psi}: \quad a_\mu = \frac{\tilde{\Psi}_{1\mu}}{\sqrt{2}}, \quad h_\mu = r^2\tilde{\Psi}_{2\mu}. \tag{A.57}$$

A.4.3 Scalar perturbation

For scalar perturbation, after fixing the gauge freedom ($h_{(LL)} = h_{(L)r} = h_{(L)\phi} = h_{r\phi} = a_{(L)} = 0$) and using the corresponding constraint equations (the (LL), $(L)r$, $(L)\phi$ and $r\phi$ components of the Einstein equation and the (L) component of the Maxwell equation), the metric and the $U(1)$ field in the linearized level are written as

$$ds_6^2 = r^2(1 + \tilde{\Phi}_2)\eta_{\mu\nu}dx^\mu dx^\nu + \left[1 + (\tilde{\Phi}_1 + \tilde{\Phi}_2) \right]\frac{dr^2}{f} + \left[1 - (\tilde{\Phi}_1 + 3\tilde{\Phi}_2) \right]f d\phi^2,$$

$$A_M dx^M = \left\{ A + \frac{1}{A'}\left[\frac{1}{2r^2}(fr^2\tilde{\Phi}_1)' + f'\tilde{\Phi}_2 \right] \right\} d\phi, \tag{A.58}$$

where $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ are functions of (x^μ, r). The bulk action is expanded up to the second order in perturbation as

$$I_6 = \frac{M_6^4}{2} \int d^6x\sqrt{-g}\left(R - 2\Lambda_6 - \frac{1}{2}F^{MN}F_{MN} \right)$$

$$= \frac{M_6^4 \Delta\phi}{2} \int d^4x L, \tag{A.59}$$

where

$$L = -\eta^{\mu\nu}\partial_\mu\tilde{\mathbf{Q}}^T\mathbf{\Omega}\partial_\nu\tilde{\mathbf{Q}} + \tilde{\mathbf{Q}}^T\mathbf{L}\tilde{\mathbf{Q}} \tag{A.60}$$

41

Here, $\mathbf{L}$ and $\mathbf{\Omega}$ are defined in (3.57) and

$$\tilde{\mathbf{Q}} = \begin{pmatrix} \tilde{Q}_1[\tilde{\Phi}] \\ \tilde{Q}_2[\tilde{\Phi}] \end{pmatrix}, \tag{A.61}$$

where

$$
\begin{aligned}
\tilde{Q}_1[\tilde{\Phi}] &\equiv \dot{\tilde{\Phi}}_2 + \frac{2f\tilde{\Phi}_1}{6f + rf'}, \\
\tilde{Q}_2[\tilde{\Phi}] &\equiv \frac{1}{A'}\left[\frac{1}{2r^2}(fr^2\tilde{\Phi}_1)' + f'\tilde{\Phi}_2\right] + \frac{frA'\tilde{\Phi}_1}{6f + rf'},
\end{aligned}
\tag{A.62}
$$

We have not written down the boundary term since it does not change the definition of the generalized Klein-Gordon norm. It is easy to check that the correct equations of motion are derived from this action.

From this form of the action we can read off the generalized Klein-Gordon norm as

$$
\begin{aligned}
(\tilde{\Phi}, \tilde{\Psi})_{KG} &\equiv -iM_6^4\Delta\phi \int d^3\mathbf{x} \int_{r_-}^{r_+} dr\, r^2 \\
&\quad \times \left[3\left(\tilde{Q}_1[\tilde{\Phi}]\partial_t\tilde{Q}_1^*[\tilde{\Psi}] - \tilde{Q}_1^*[\tilde{\Psi}]\partial_t\tilde{Q}_1[\tilde{\Phi}]\right) + \frac{1}{f}\left(\tilde{Q}_2[\tilde{\Phi}]\partial_t\tilde{Q}_2^*[\tilde{\Psi}] - \tilde{Q}_2^*[\tilde{\Psi}]\partial_t\tilde{Q}_2[\tilde{\Phi}]\right)\right]
\end{aligned}
\tag{A.63}
$$

where the solutions $\tilde{\Phi}$ and $\tilde{\Psi}$ are specified by pairs of five-dimensional functions $(\tilde{\Phi}_1, \tilde{\Phi}_2)$ and $(\tilde{\Psi}_1, \tilde{\Psi}_2)$, respectively. For $\tilde{\Phi}_i = \Phi_i(r)e^{i\eta_{\mu\nu}k_1^\mu x^\nu}$ and $\tilde{\Psi}_i = \Psi_i(r)e^{i\eta_{\mu\nu}k_2^\mu x^\nu}$, by using the equations of motion, integrating by part and using $f(r_\pm) = 0$, we obtain

$$
\begin{aligned}
(\tilde{\Phi}, \tilde{\Psi})_{KG} &\equiv M_6^4\Delta\phi(2\pi)^3(\omega_1 + \omega_2)\delta^3(\mathbf{k}_1 - \mathbf{k}_2)e^{-i(\omega_1-\omega_2)t}\left\{\int_{r_-}^{r_+} \frac{dr}{8(2rf' + 6f + \Lambda_6 r^2)}\right. \\
&\quad \times \left[\frac{1}{2}(m_1^2 + m_2^2)r^2\Phi_1\Psi_1^* + 2r^4f'(\Psi_2^*\partial_r\Phi_1 + \Phi_2\partial_r\Psi_1^*)\right. \\
&\quad + r^2(13f + 3\Lambda_6 r^2)\Phi_1\Psi_1^* + 4r^2(2rf' + 12f + 3\Lambda_6 r^2)\Phi_2\Psi_2^* \\
&\quad \left.+ \frac{r^2}{f}(r^2f'^2 + 10rff' + 24f^2 + 3\Lambda_6 r^2 f)(\Phi_1 + 2\Phi_2)(\Psi_1^* + 2\Psi_2^*)\right] \\
&\quad \left.+ \left[\frac{3r^3f'\Phi_1\Psi_1^*}{16(2f' + \Lambda_6 r)}\right]_{r=r_-}^{r=r_+}\right\},
\end{aligned}
\tag{A.64}
$$

where $\omega_i = k_i^0$ and $m_i^2 = -\eta_{\mu\nu}k_i^\mu k_i^\nu$. It is shown that $(\tilde{\Phi}, \tilde{\Psi})_{KG} = 0$ for $m_1^2 \neq m_2^2$ and that $(\tilde{\Phi}, \tilde{\Psi})_{KG}$ is time independent.

42

References

[1] S. Perlmutter, et al., Astrophys. J. **517**, 565 (1999).

[2] B. P. Schmidt, et al., Astrophys. J. **507**, 46 (1998).

[3] A. G. Riess et al., Astron. J. **116**, 1009 (1998).

[4] S. Kachru, R. Kallosh, A. Linde and S. P. Trivedi, Phys. Rev. **D68**, 046005 (2003) [hep-th/0301240].

[5] C. Escoda, M. Gomez-Reino and F. Quevedo, JHEP **0311**, 065 (2003) [hep-th/0307160].

[6] C. P. Burgess, R. Kallosh and F. Quevedo JHEP **0310**, 056 (2003) [hep-th/0309187].

[7] S. Kachru, R. Kallosh, A. Linde, J. Maldacena, L. McAllister, S. P. Trivedi, JCAP **0310**, 013 (2003) [hep-th/0308055].

[8] J. J. Blanco-Pillado, C. P. Burgess, J. M. Cline, C. Escoda, M. Gomez-Reino, R. Kallosh, A. Linde and F. Quevedo, JHEP **0411**, 063 (2004) [hep-th/0406230].

[9] A. Buchel and Radu Roiban, Phys. Lett. **B590**, 284 (2004) [hep-th/0311154]; M. Berg and M. Haack, Phys. Rev. **D71**, 026005 (2005) [hep-th/0404087]; C. P. Burgess, J. M. Cline, H. Stoica and F. Quevedo, JHEP **0409**, 033 (2004) [hep-th/0403119]; O. DeWolfe, S. Kachru and H. Verlinde, JHEP **0405**, 017 (2004) [hep-th/0403123]; N. Iizuka and S. P. Trivedi, Phys. Rev. **D70**, 043519 (2004) [hep-th/0403203]; A. Buchel and A. Ghodsi, Phys. Rev. **D70**, 126008 (2004) [hep-th/0404151]; H. Firouzjahi and S. H. Tye, JCAP **0503**, 009 (2005) [hep-th/0501099]; J. P. Hsu, R. Kallosh and S. Prokushkin, JCAP **0312**, 009 (2003) [hep-th/0311077]; H. Firouzjahi and S. H. Tye, Phys. Lett. **B584**, 147 (2004) [hep-th/0312020]; J. P. Hsu and R. Kallosh, JHEP **0404**, 042 (2004) [hep-th/0402047]; F. Koyama, Y. Tachikawa and T. Watari, Phys. Rev. **D69**, 106001 (2004), Erratum-ibid. **D70**, 129907 (2004) [hep-th/0311191]; L. McAllister, JCAP **0602**, 010 (2006) [hep-th/0502001]; K. Dasgupta, J. P. Hsu, R. Kallosh, A. Linde and M. Zagermann, JHEP **0408**, 030 (2004) [hep-th/0405247]; P. Chen, K. Dasgupta, K. Narayan, M. Shmakova and M. Zagermann, JHEP **0509**, 009 (2005) [hep-th/0501185]; Yun-Song Piao, Phys. Rev. **D71**, 107301 (2005) [gr-qc/0502006]; D. Cremades, F. Quevedo and A. Sinha, JHEP **0510**, 106 (2005) [hep-th/0505252].

[10] J. Polchinski, "Introduction to Cosmic F- and D-strings", hep-th/0412244 and references therein.

[11] S. Mukohyama, Phys. Rev. **D72**, 061901 (2005) [hep-th/0505042].

[12] S. Mukohyama, Y. Sendouda, H. Yoshiguchi and S. Kinoshita, JCAP **0507**, 013 (2005) [hep-th/0506050].

[13] L. Randall and R. Sundrum, Phys. Rev. Lett. **83**, 3370 (1999) [hep-ph/9905221].

[14] W. D. Goldberger and M. B. Wise, Phys. Rev. Lett. **83**, 4922 (1999) [hep-th/0007065].

[15] T. Tanaka and X. Montes, Nucl. Phys. **B582**, 259 (2000) [hep-th/0001092].

[16] C. Csaki, M. L. Graesser and G. D. Kribs, Phys. Rev. **D63**, 065002 (2001) [hep-th/0008151].

[17] S. Mukohyama and L. Kofman, Phys. Rev. **D65**, 124025 (2002) [hep-th/0112115].

[18] S. Mukohyama, Phys. Rev. **D65**, 084036 (2002) [hep-th/0112205].

[19] J. Louko and D. L. Wiltshire, JHEP **0202**, 007 (2002) [hep-th/0109099]

[20] S. M. Carroll and M. M. Guica, "Sidestepping the Cosmological Constant with Football Shaped Extra Dimensions", hep-th/0302067.

[21] J. Garriga and M. Porrati, JHEP **0408**, 028 (2004) [hep-th/0406158].

[22] I. Navarro, Class. Quant. Grav. **20**, 3603 (2003) [hep-th/0305014]; JCAP **0309**, 004 (2003) [hep-th/0302129].

[23] J. Vinet and J. M. Cline, Phys. Rev.**D71**, 064011 (2005) [arXiv:hep-th/0501098]; Phys. Rev. **D70**, 083514 (2004) [arXiv:hep-th/0406141].

[24] J. M. Cline, J. Descheneau, M. Giovannini and J. Vinet, JHEP **0306**, 048 (2003) [arXiv:hep-th/0304147].

[25] C. de Rham and A. J. Tolley, hep-th/0511138.

[26] H. M. Lee and A. Papazoglou, hep-th/0602208.

[27] V. P. Frolov, W. Israel and W. G. Unruh, Phys. Rev. **D39**, 1084 (1989).

[28] Y. Sendouda, S. Mukohyama, S. Kinoshita and H. Yoshiguchi: in preperation.

[29] S. Kinoshita, S. Mukohyama, Y. Sendouda and H. Yoshiguchi: in progress.

[30] G. W. Gibbons and D. L. Wiltshire, Nucl. Phys. **B287**, 717 (1987).

[31] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical recipes in C*, 1992 (Cambridge University Press).

[32] S. Mukohyama, Phys. Rev. **D61**, 124021 (2000) [e-Print Archive: gr-qc/9910013].